Exercise Set 1

If you need credit for the course, please turn in half of the following exercises (of a choice of your own).

1. (a) Describe the Zariski topology on the affine line $\mathbf{A}^1$.
   (b) The Euclidean topology on $\mathbb{C}^n$ is finer than the Zariski topology.
   (c) If we identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$. Which one is finer?

2. (a) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain.
   (b) Let A and B be $\mathbf{k}$-algebras of finite type. If A and B are reduced (resp. integral domains), then the same holds for $A \otimes_{\mathbf{k}} B$. Show that the assumption that A and B are of finite type can be omitted.

3. Let $\phi : (X, A) \to (Y, B)$ be a morphism of affine varieties and let $\phi^* : B \to A$ be the associated comorphism. Prove the following.
   (a) If ϕ^* is surjective, then ϕ maps X onto a closed subset of Y;
   (b) ϕ^* is injective if and only if ϕX is dense in Y.

4. Let (X, A) be an affine variety and $Y \subset X$. Show that $(Y, A|_Y)$ is a subvariety of (X, A) if and only if Y is closed in X.

5. (a) Let Y be the plane curve $y = x^2$. Show that $\mathbf{k}[Y]$ is isomorphic to a polynomial ring in one variable.
   (b) Let Z be the plane curve $xy = 1$. Show that $\mathbf{k}[Z]$ is not isomorphic to a polynomial ring in one variable.

6. Show that a morphism of affine varieties $\phi : X \to Y$ is continuous for the Zariski topology.

7. A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.
   (a) Let $\phi : \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that the map ϕ defines a bijective bicontinuous morphism of $\mathbf{A}^1$ onto the curve $y^2 = x^3$, but that ϕ is not an isomorphism.
   (b) Let the characteristic of the base field $\mathbf{k}$ be $p > 0$, and define a map $\phi : \mathbf{A}^1 \to \mathbf{A}^1$ by $t \mapsto t^p$. Show that ϕ is bijective and bicontinuous but not an isomorphism. This is called the Frobenius morphism.

8. Let Y be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.