Introduction to Coalescent theory

Matthieu Foll

22.11.2011

Population Genomics course, Helsinki

Introduction to Coalescent Theory

Classical population genetics theory tries to predict what will happen in the future of a given population. It is a **prospective** approach.

Coalescent theory is a **retrospective approach** to population genetics based on the genealogy of gene copies.

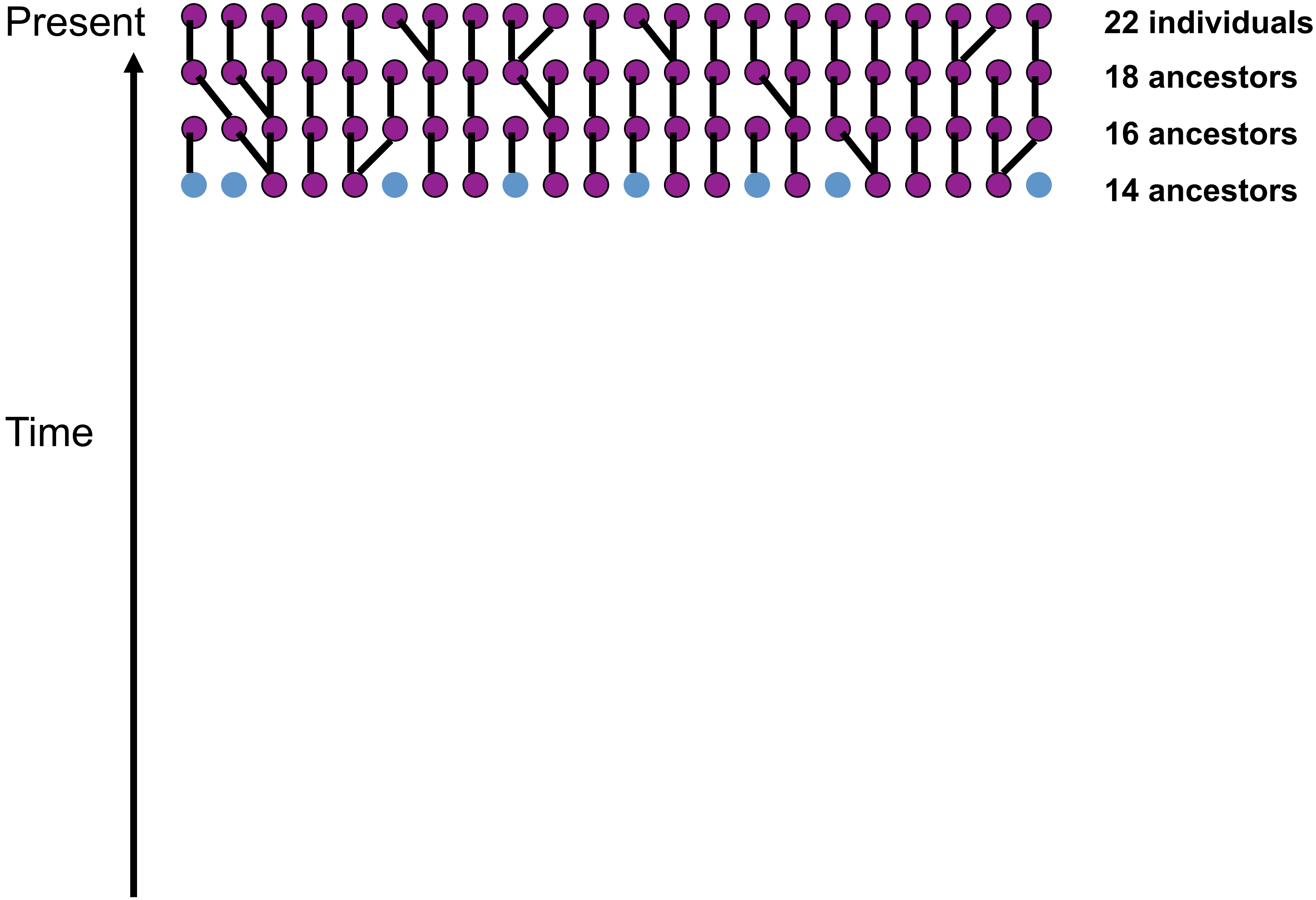
It uses mathematics for describing the characteristics of the joining of lineages **back in time** to a **common ancestor**.

This lineage joining is referred to as **coalescence**.


Present
22 individuals
Time

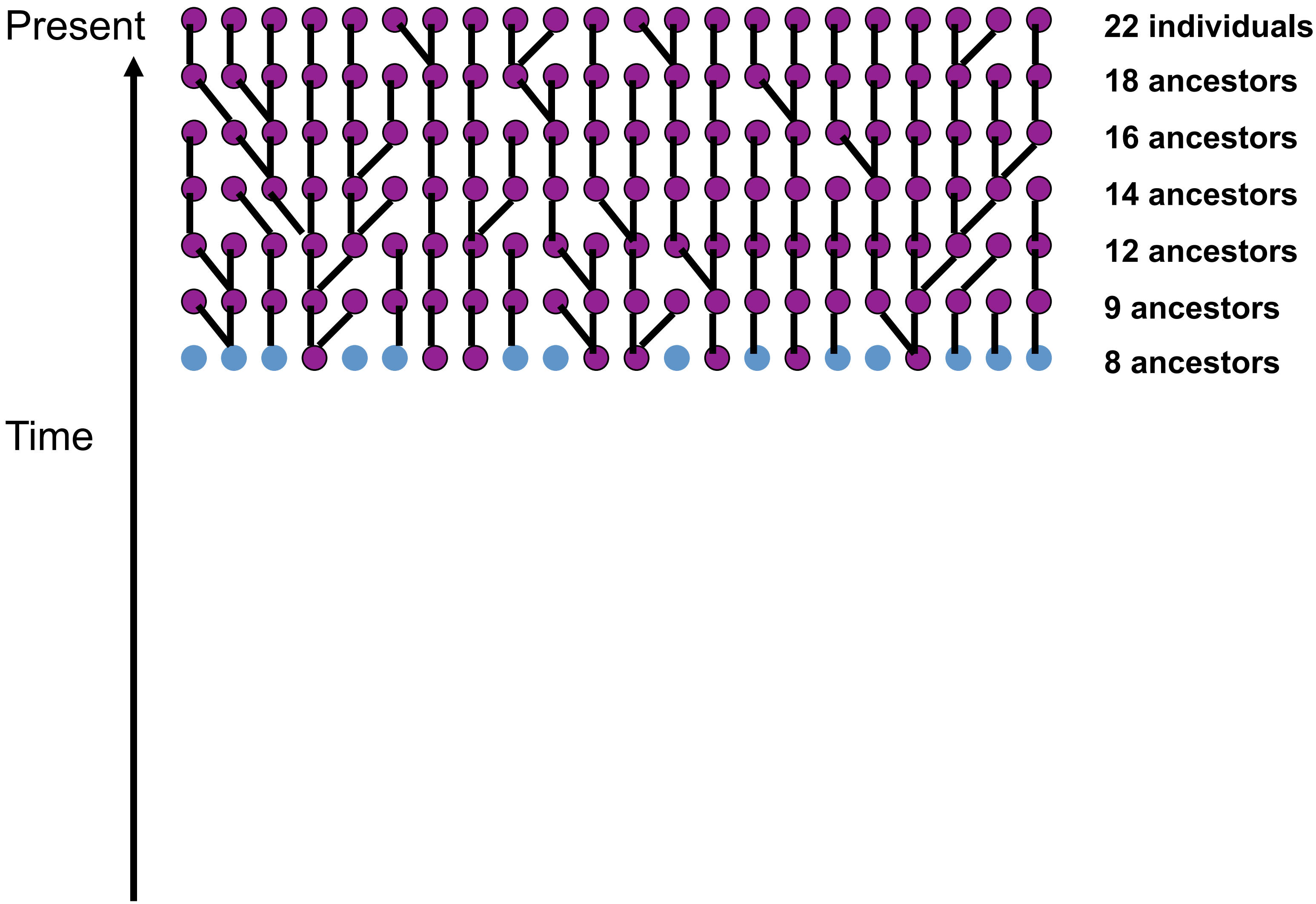

Present
22 individuals
18 ancestors
Time


Present
Time
22 individuals
18 ancestors
16 ancestors

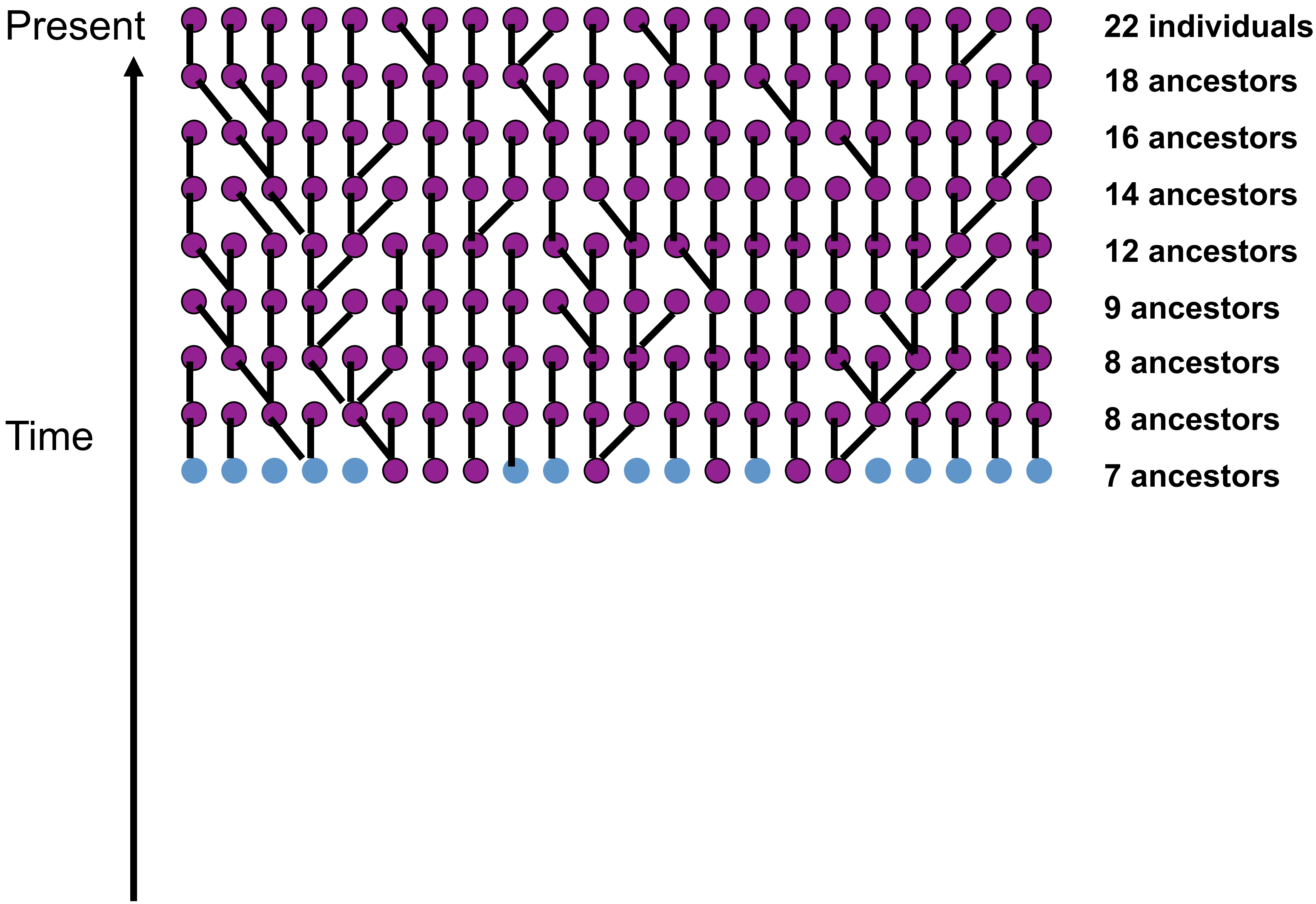

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors

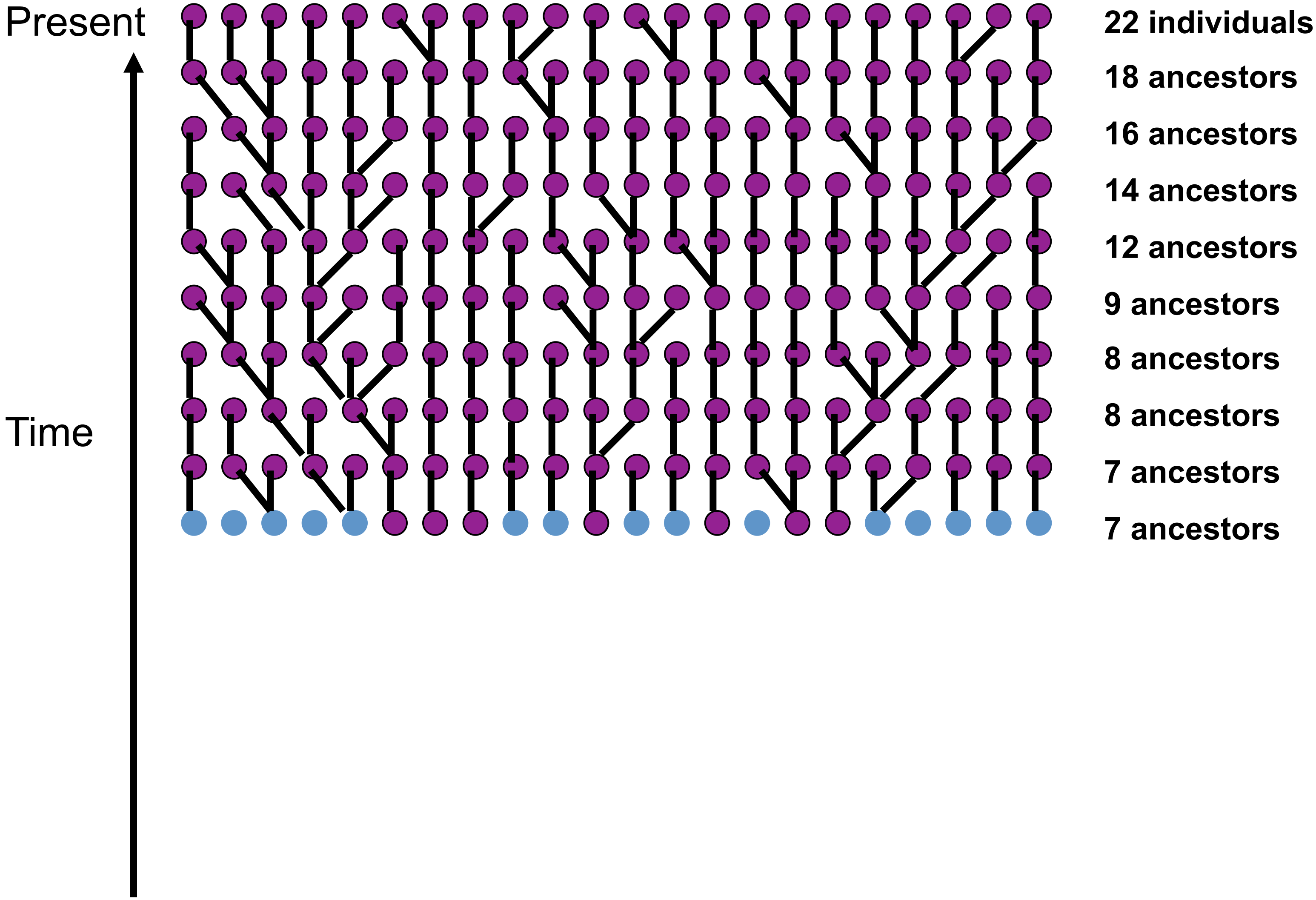

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors


Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors

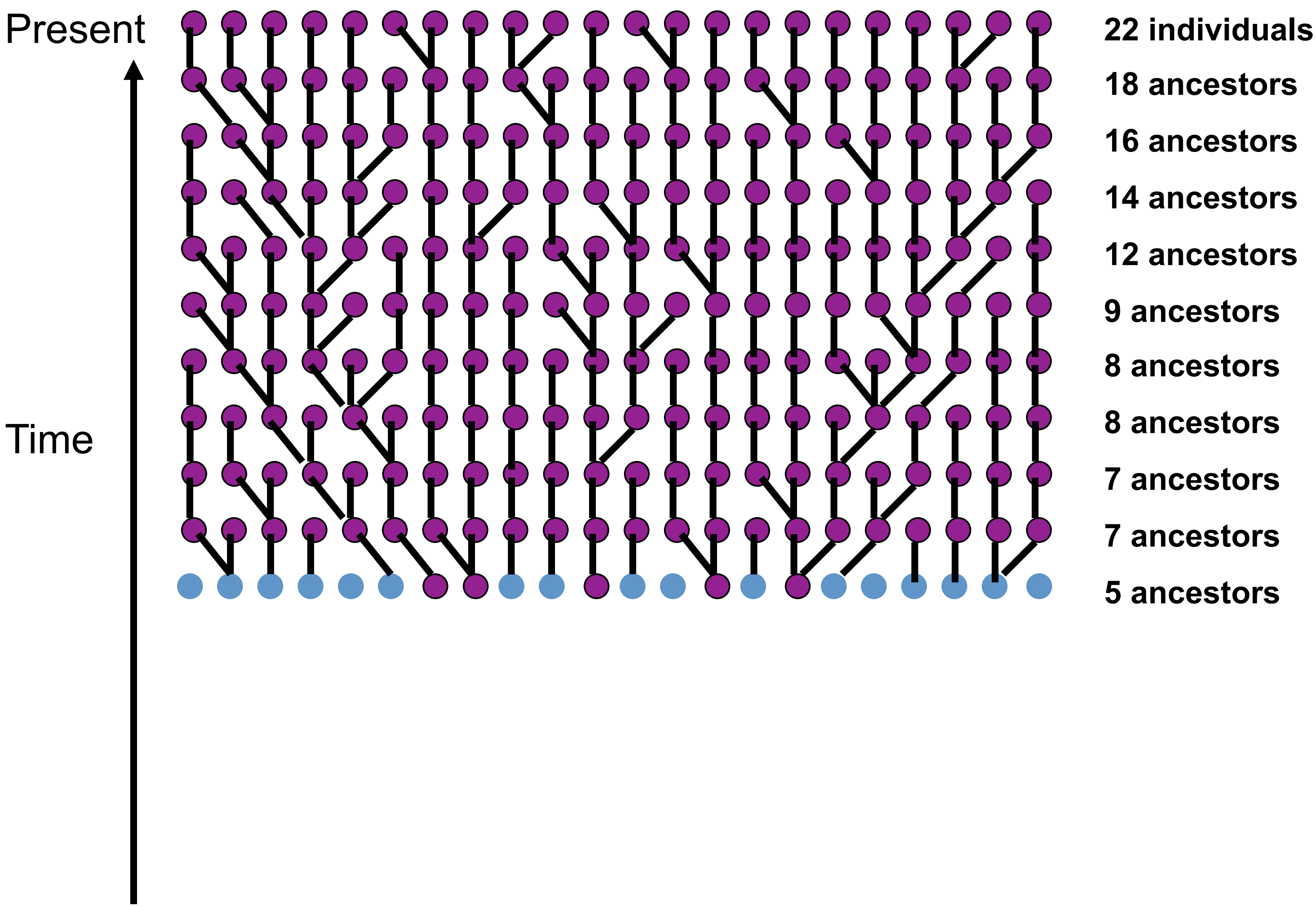

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors

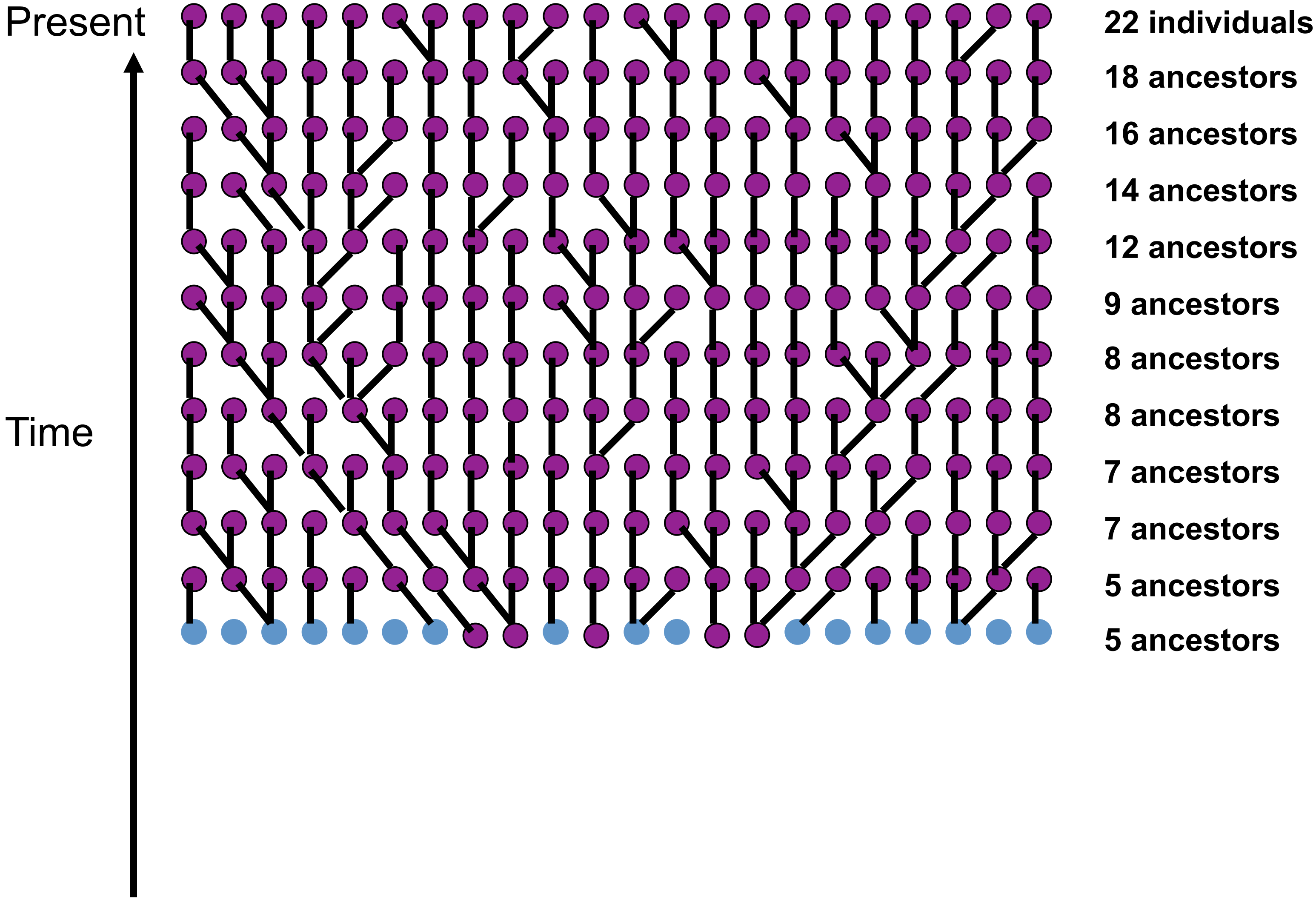

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors

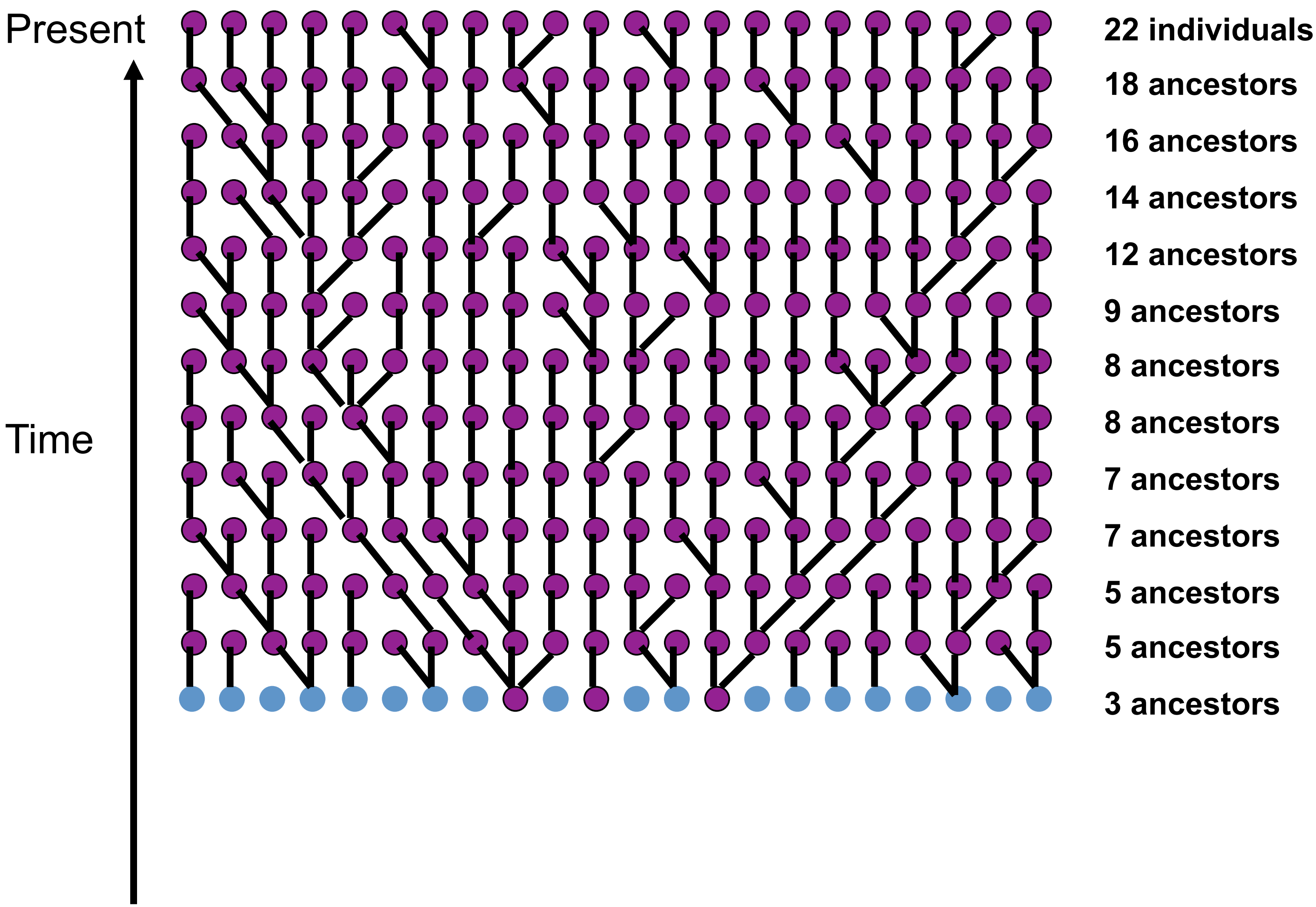

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors

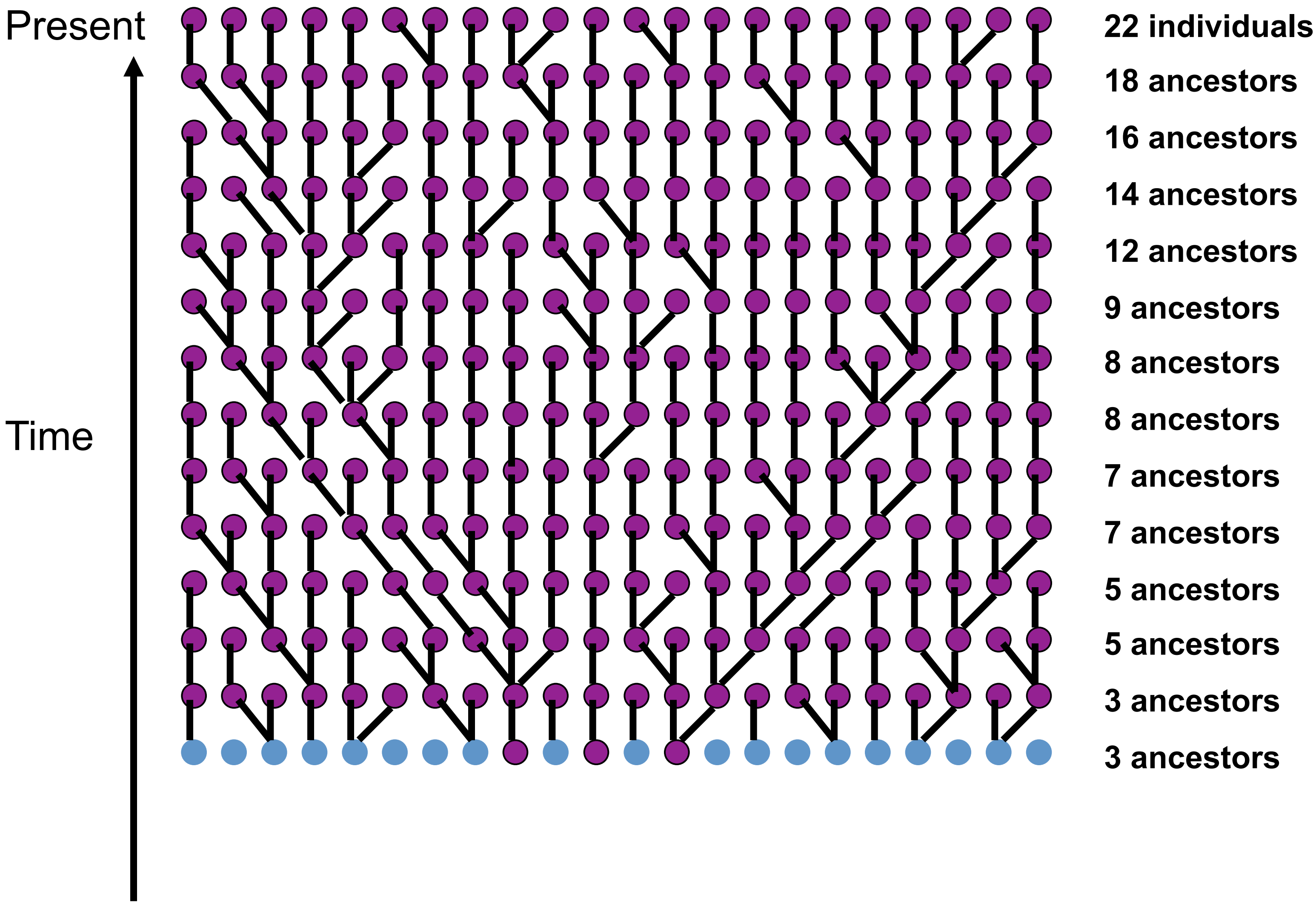


Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors

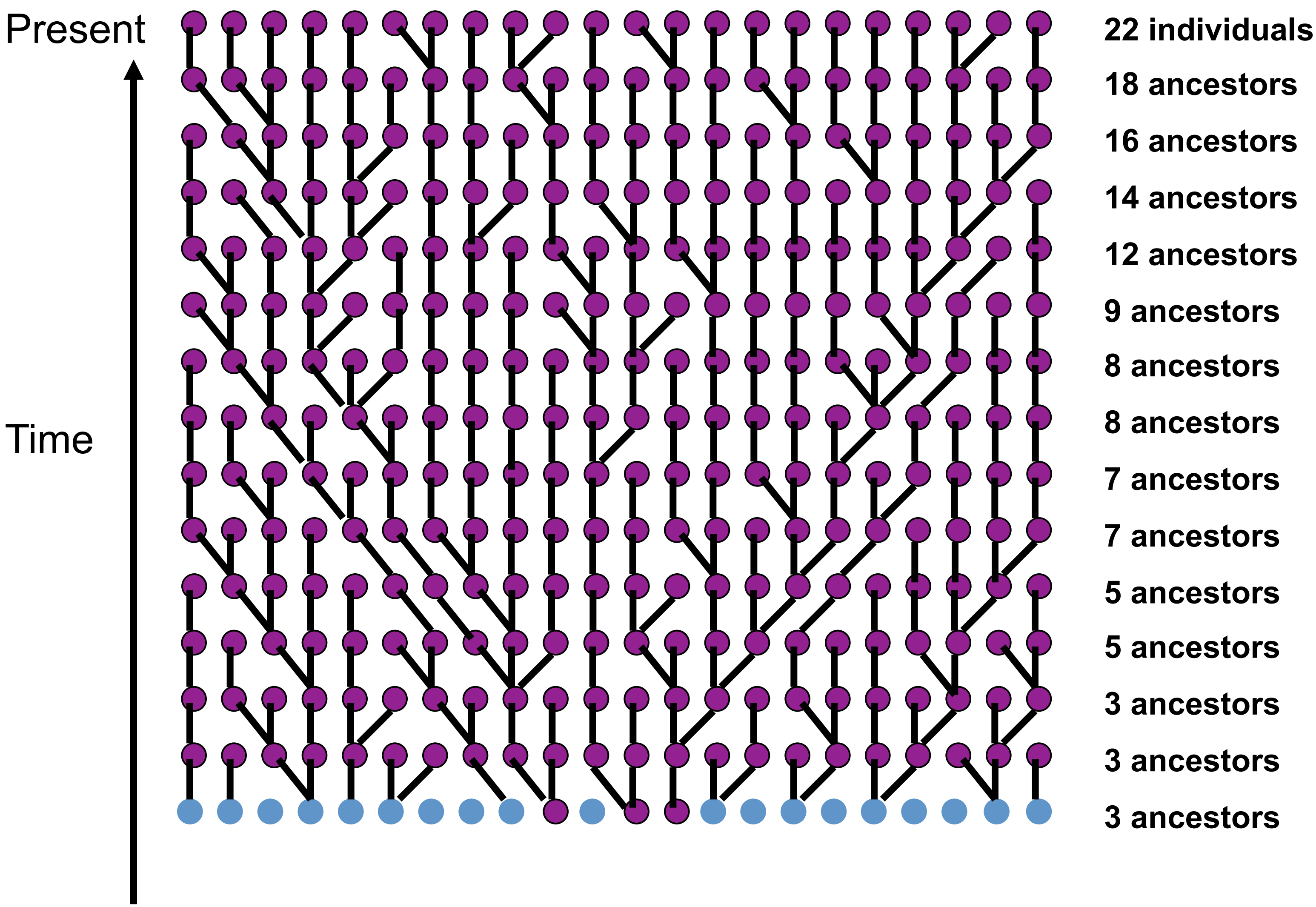

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors

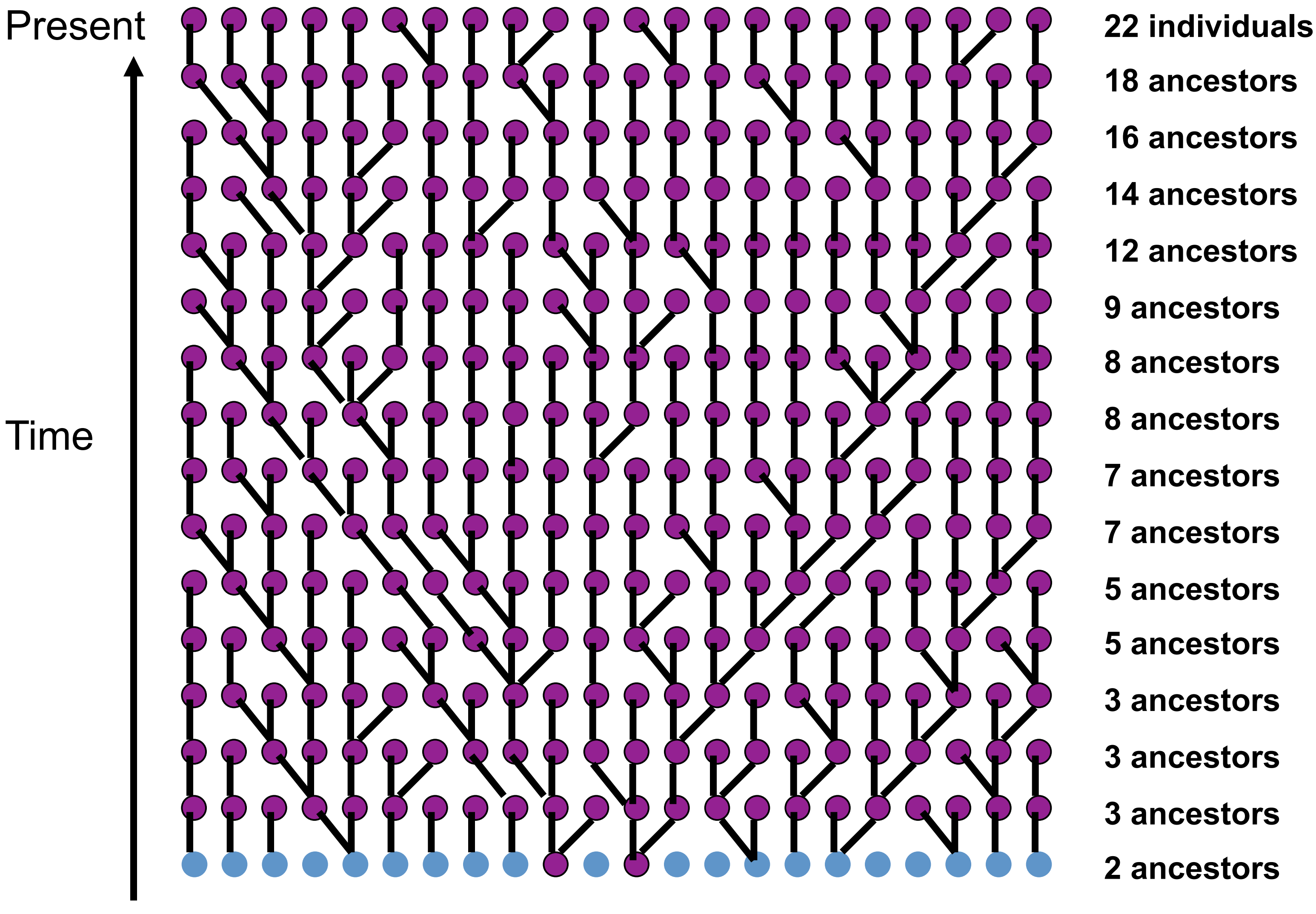

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors
5 ancestors

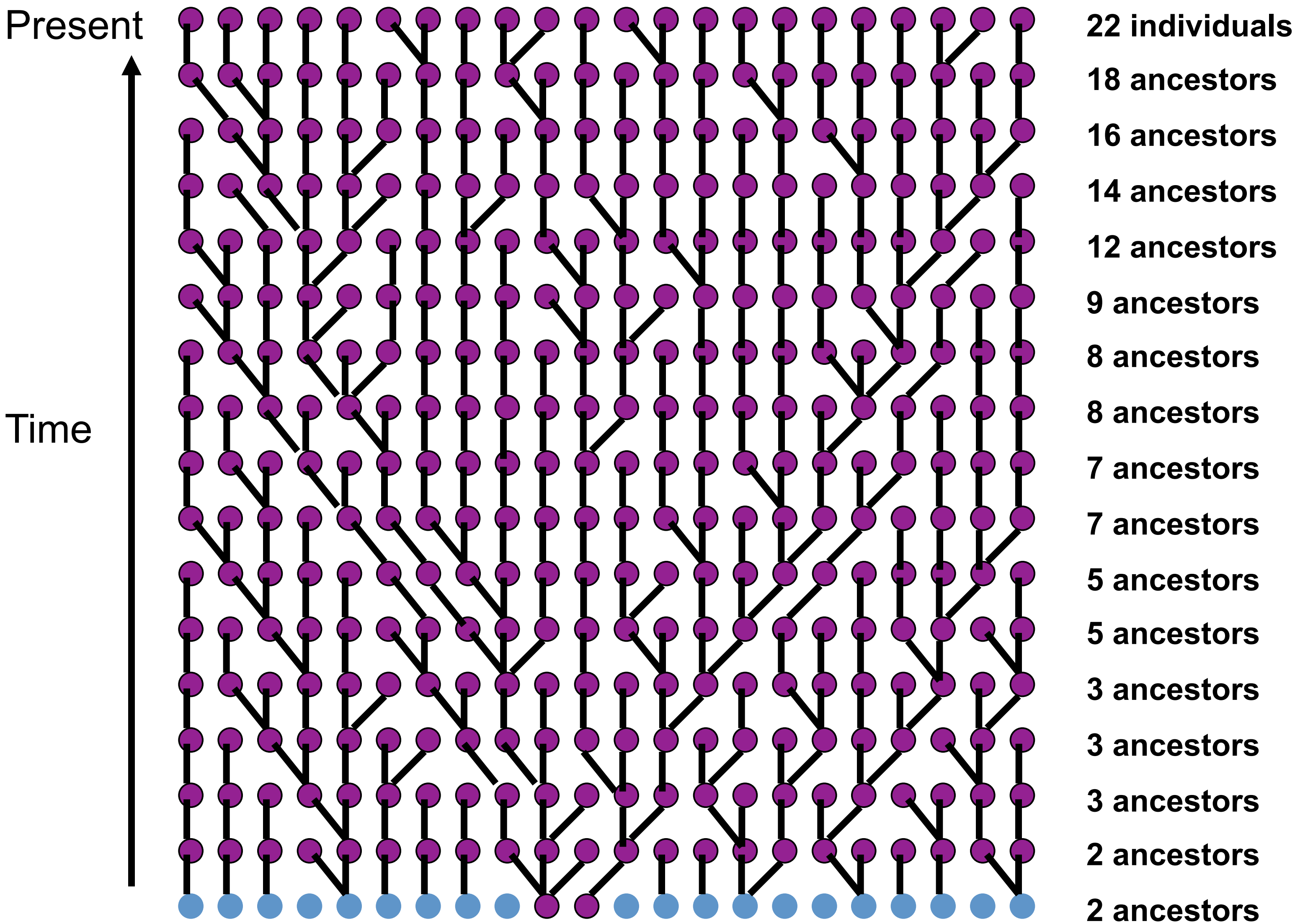

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors
5 ancestors
3 ancestors

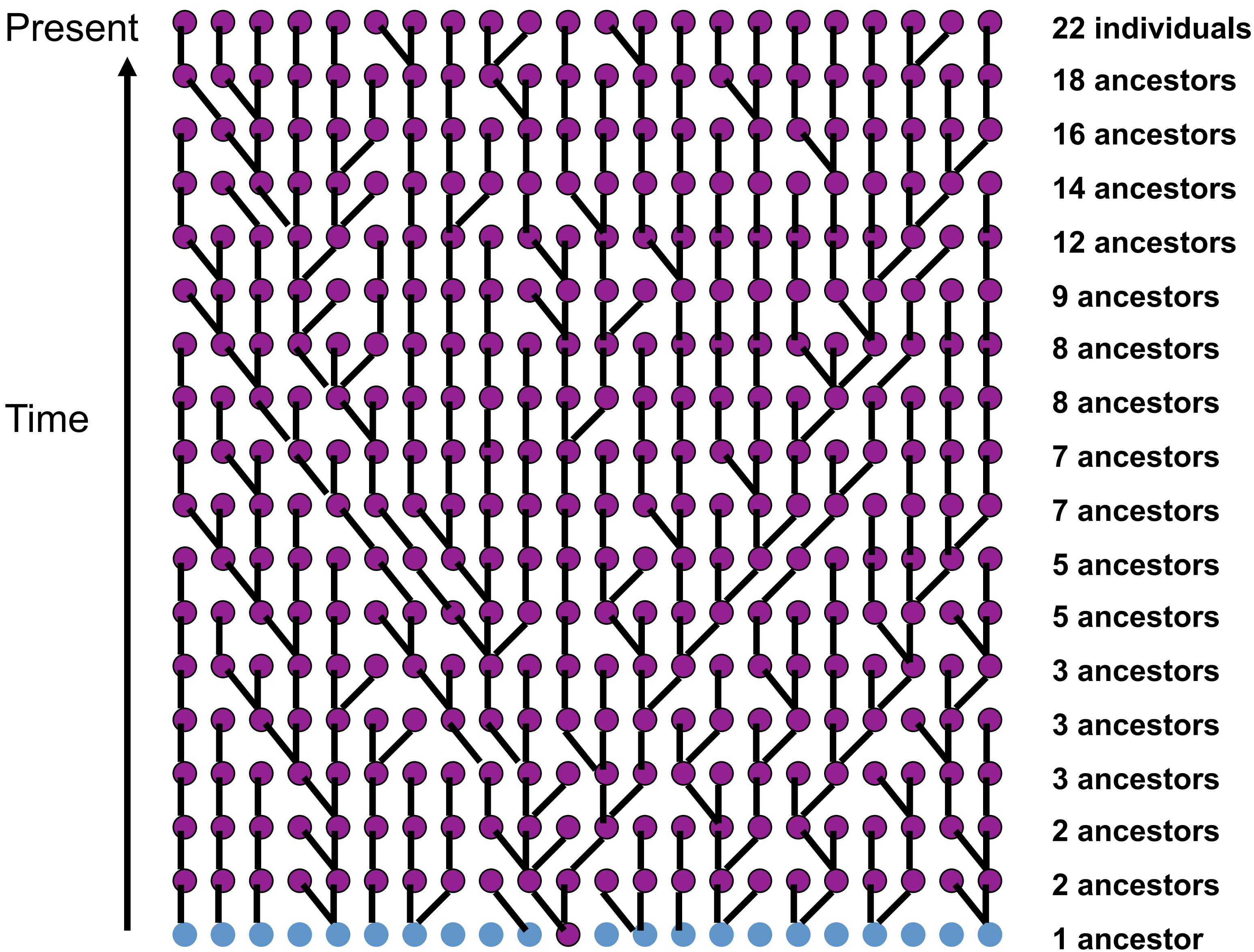


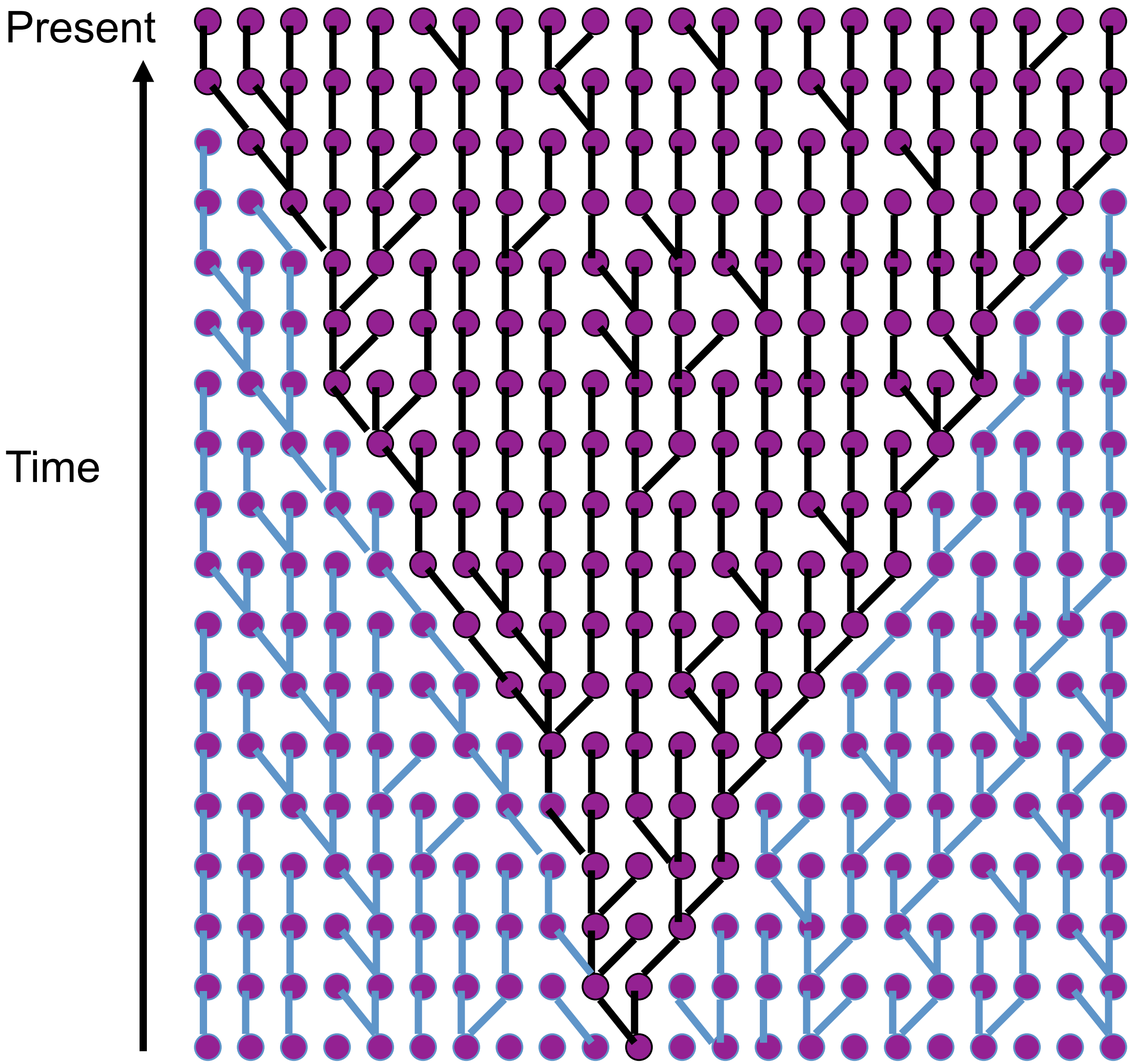
Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors
5 ancestors
3 ancestors
3 ancestors

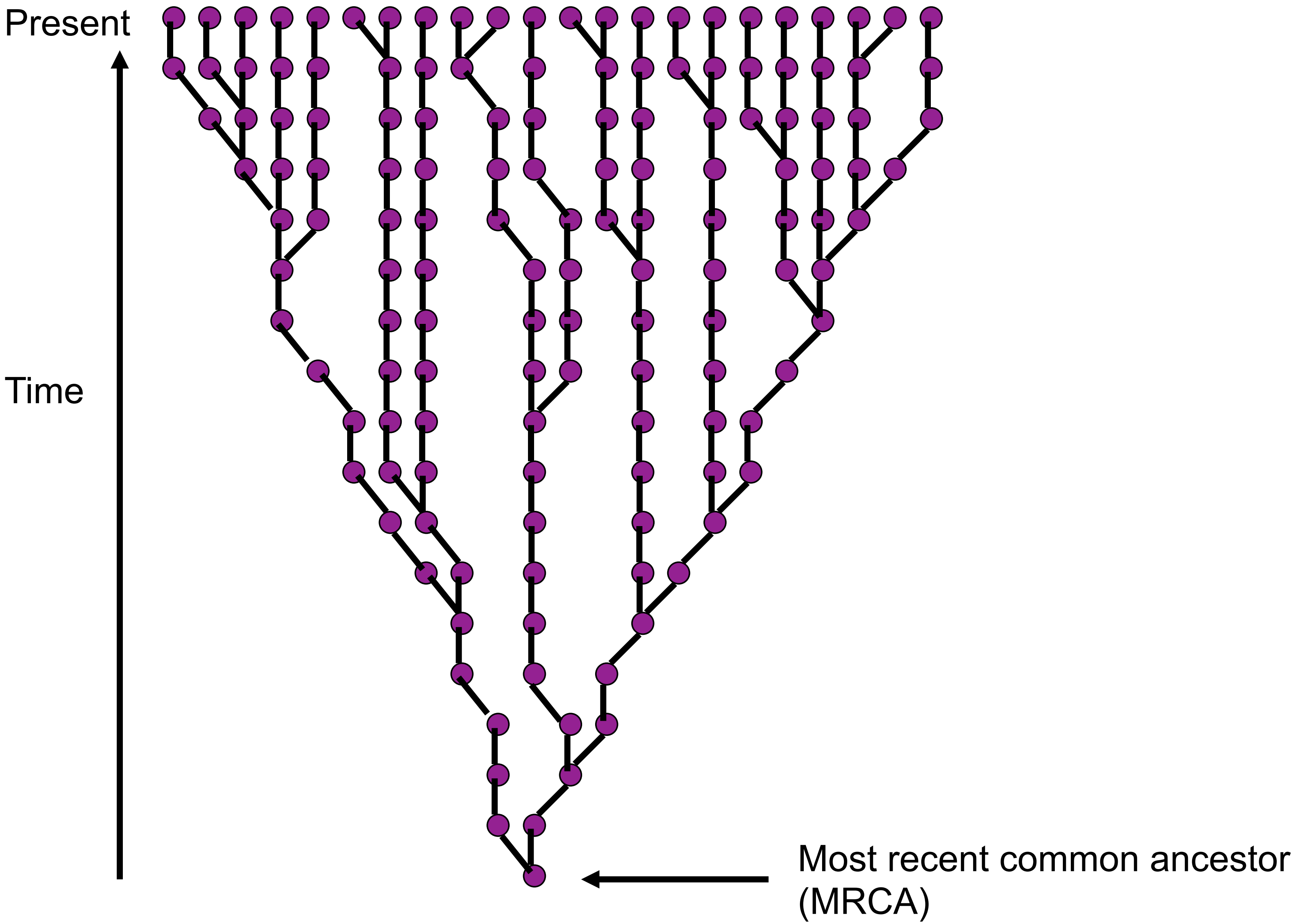

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors
5 ancestors
3 ancestors
3 ancestors
3 ancestors


Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors
5 ancestors
3 ancestors
3 ancestors
3 ancestors
2 ancestors


Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors
5 ancestors
3 ancestors
3 ancestors
3 ancestors
2 ancestors
2 ancestors

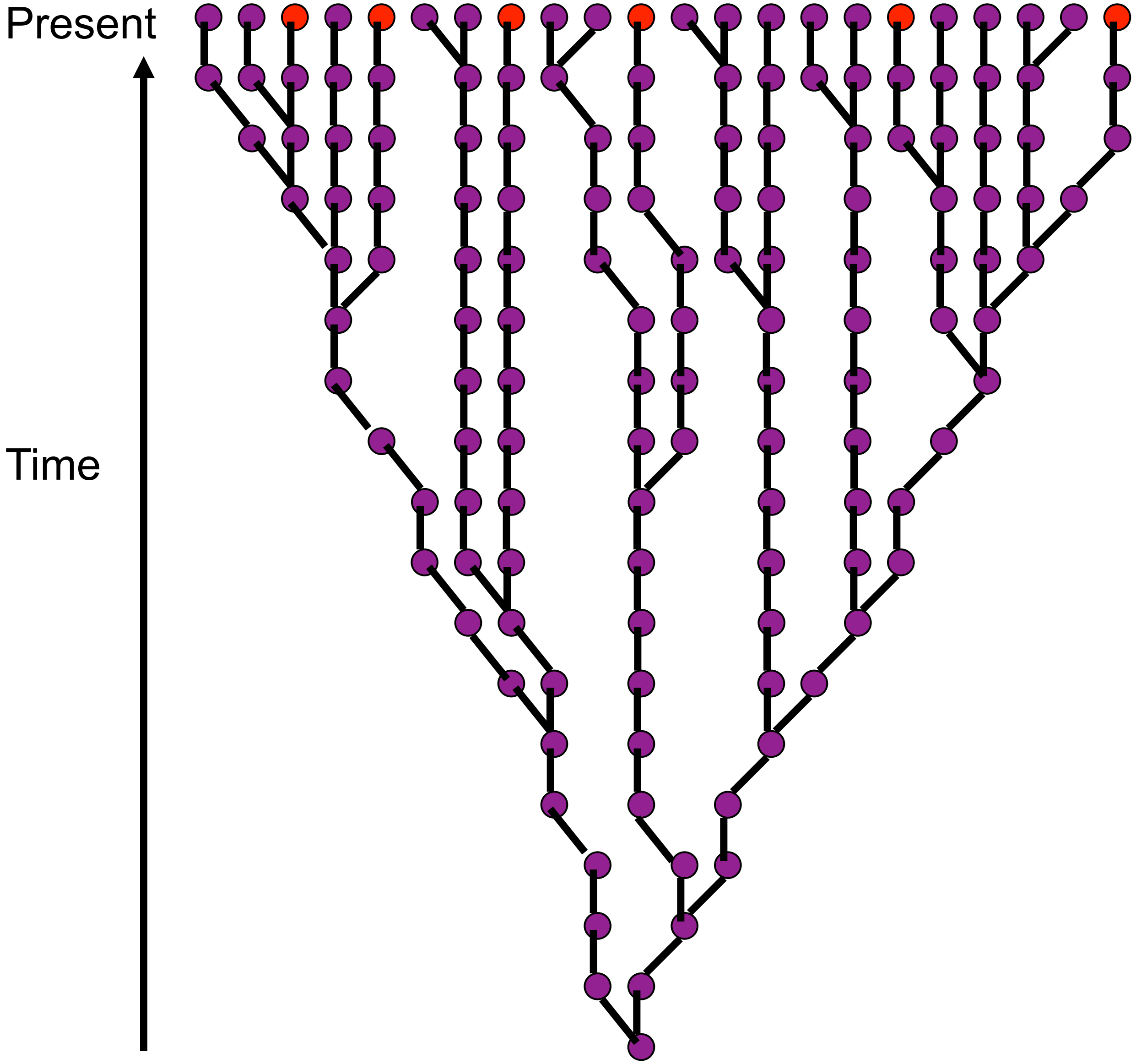

Present
Time
22 individuals
18 ancestors
16 ancestors
14 ancestors
12 ancestors
9 ancestors
8 ancestors
8 ancestors
7 ancestors
7 ancestors
5 ancestors
5 ancestors
3 ancestors
3 ancestors
3 ancestors
2 ancestors
2 ancestors
1 ancestor

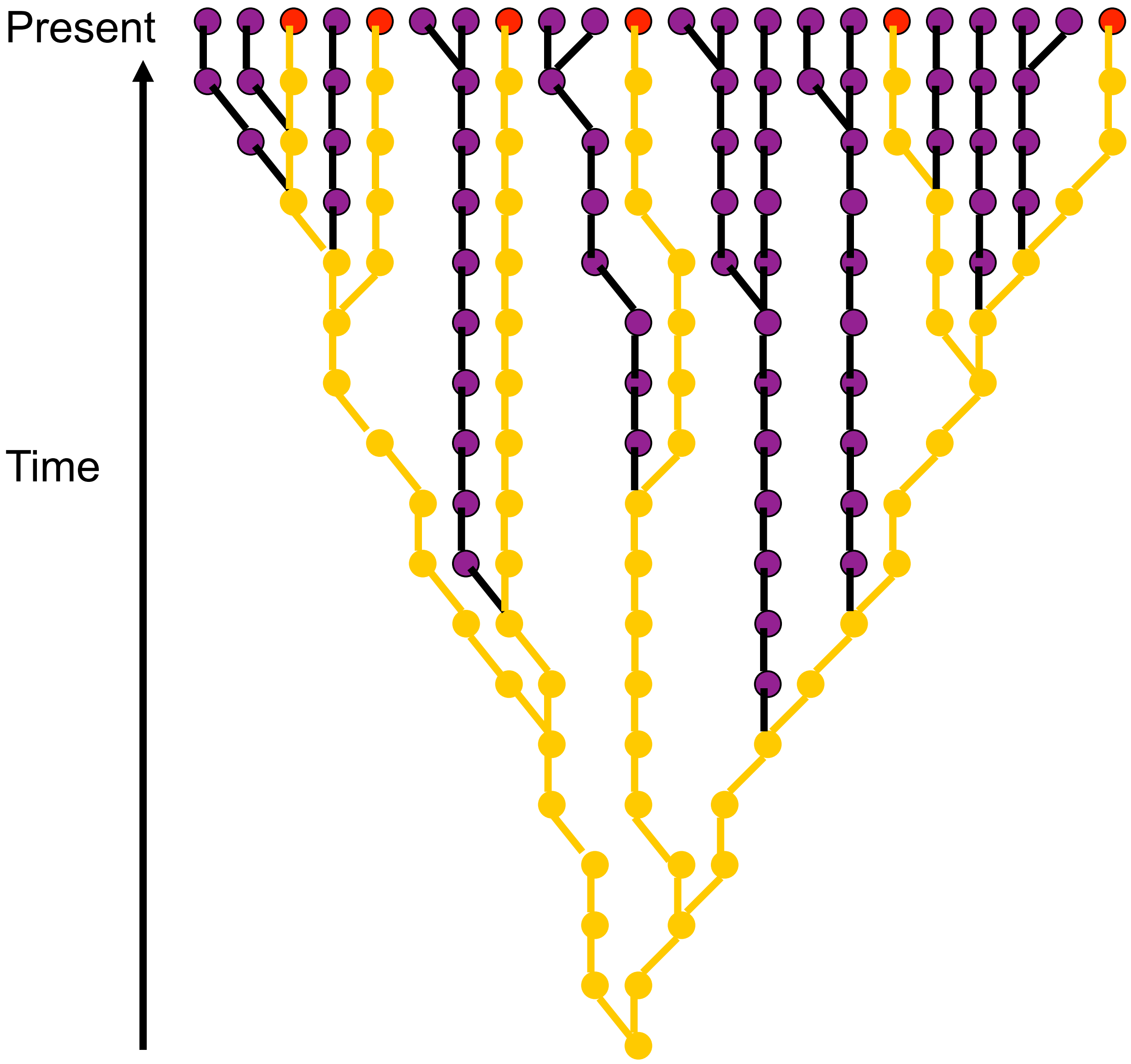

Present
Time

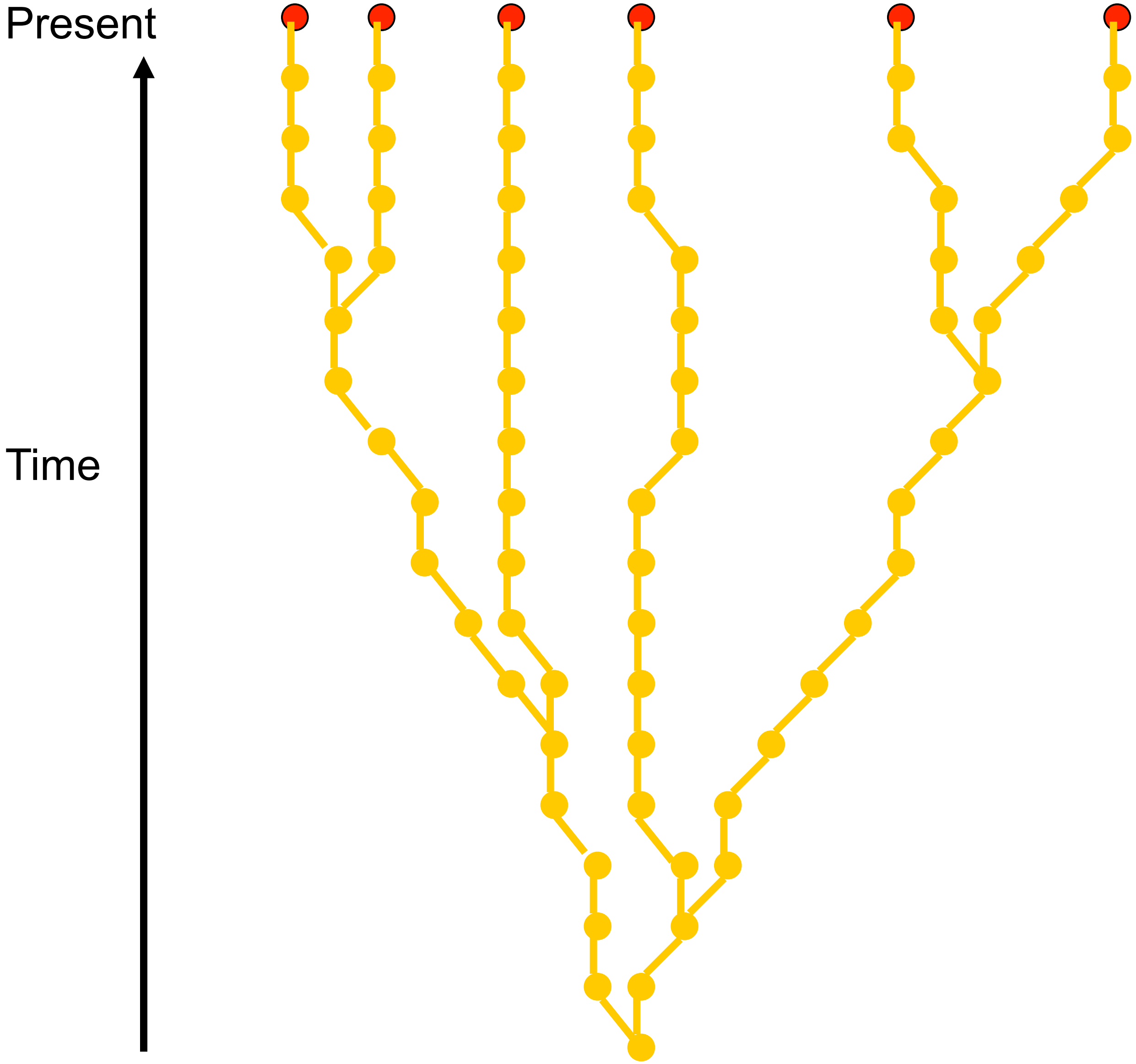


Present
Time
Most recent common ancestor
(MRCA)

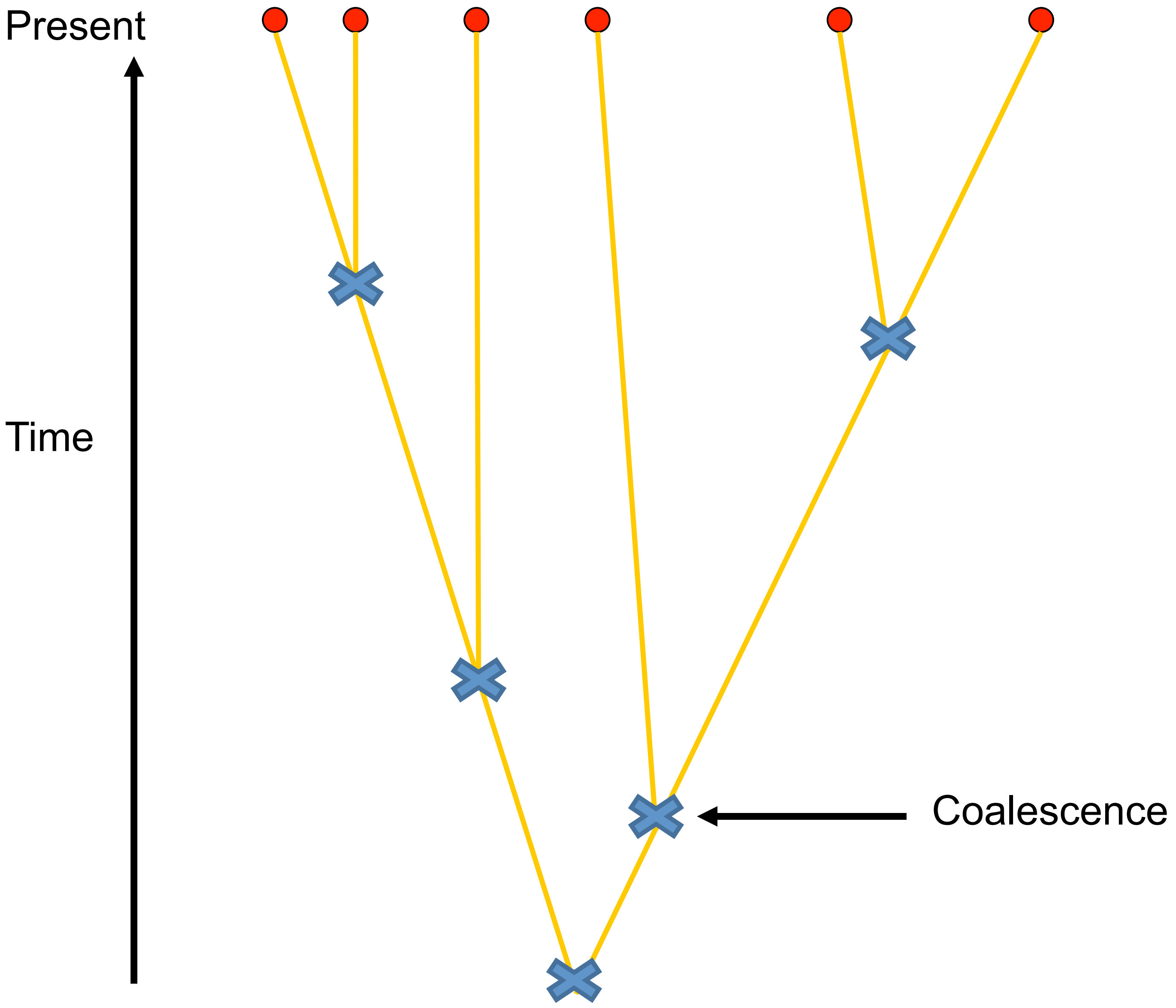
- Most of the time, we are interested in the genealogy of a sample taken from the whole population.


Present
Time


Present
Time

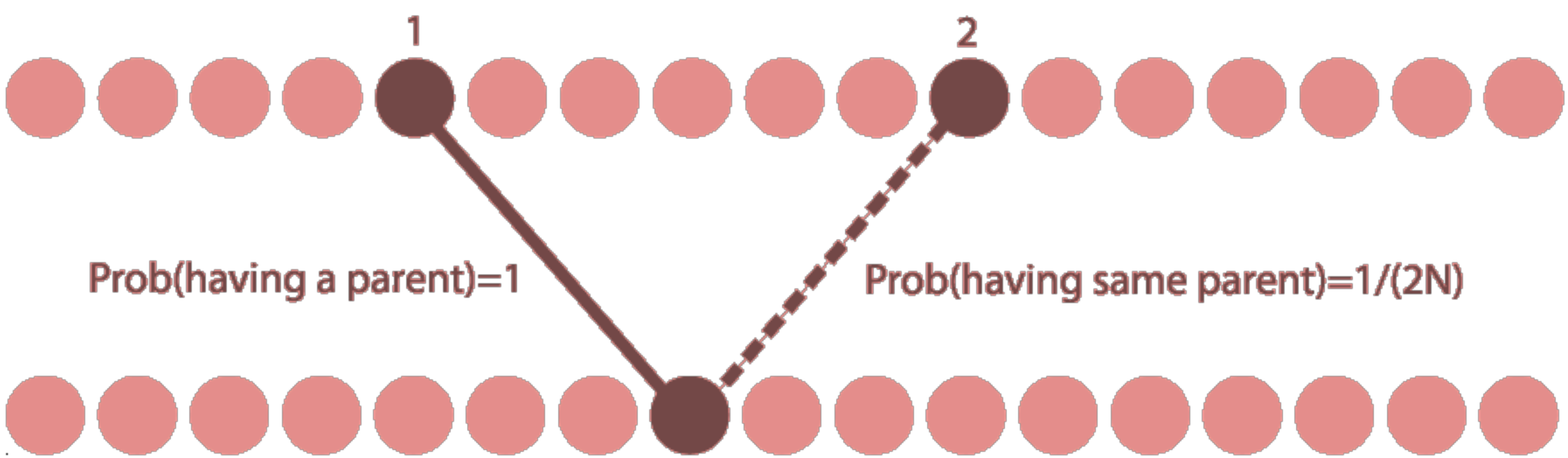


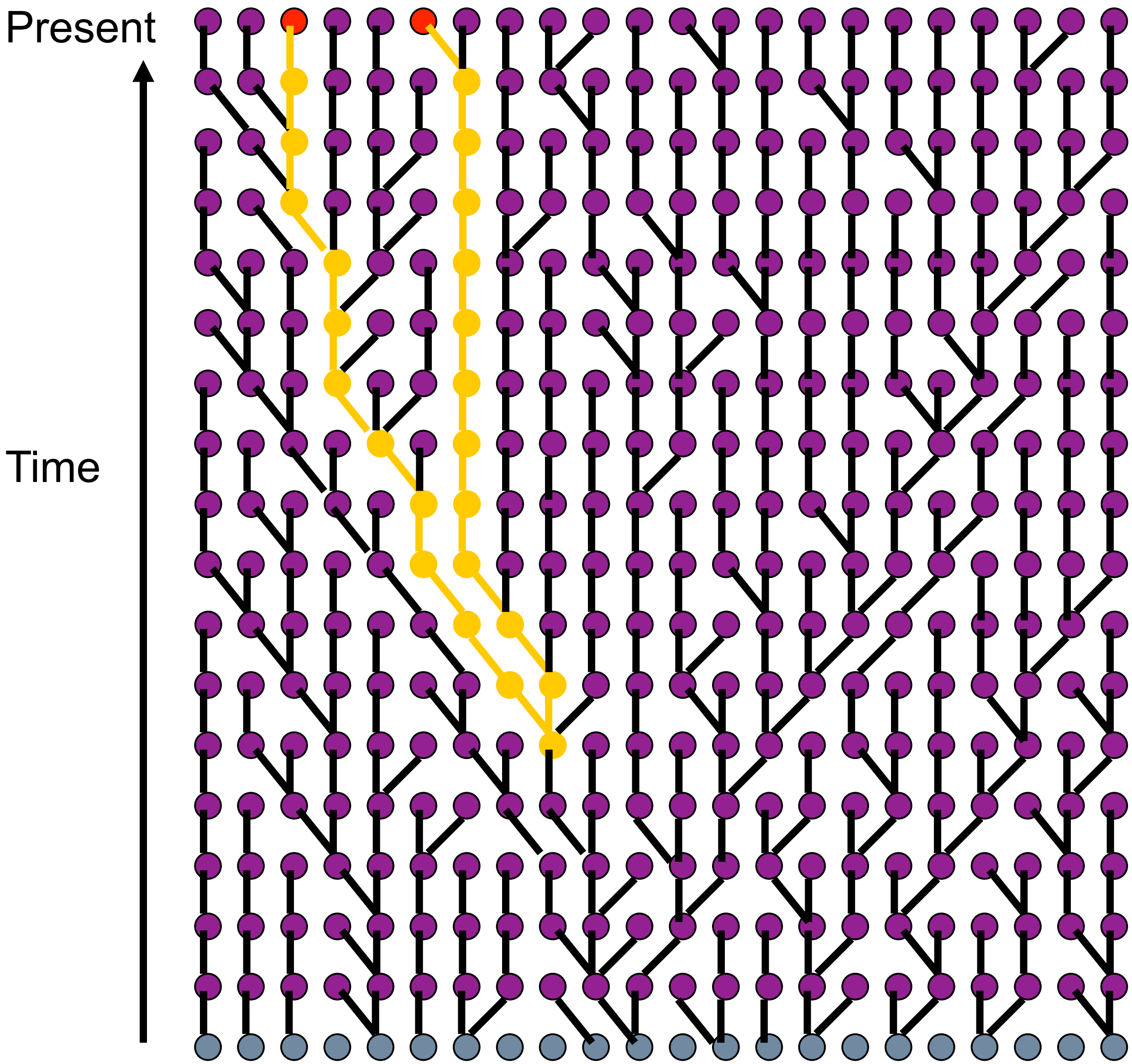
Present
Time


Present
Time
Coalescence

Wright-Fisher **demographic** model

- Forwards-in-time model of a population in a **constant-size**, **random-mating**, evolving in **discrete** generations (non-overlapping)





$$P = 1 - \frac{1}{2N}$$

$$P = 1 - \frac{1}{2N}$$

$$P = \frac{1}{2N}$$

$$P = \left(1 - \frac{1}{2N}\right)^{11} * \frac{1}{2N}$$

Wright-Fisher model

- The time to coalesce for two genes follows a **geometric distribution** with parameter 1/2N
- The probability that two genes coalesce t generations ago is given by:

$$\left(1-\frac{1}{2N}\right)^{t-1} * \frac{1}{2N}$$

- The **expected time** to coalesce is **2N**
- But the **variance is big**: 2N*(2N-1)~$\mathbf{N^2}$

Kingman's "n-coalescent"

- We now consider k genes
- There is more chance to observe a coalescent event: 1/2N for each possible pair among the k
- Number of possible pairs: $\binom{k}{2} = \frac{k*(k-1)}{2}$
- The total probability of any one pair to coalesce in the former generation is then

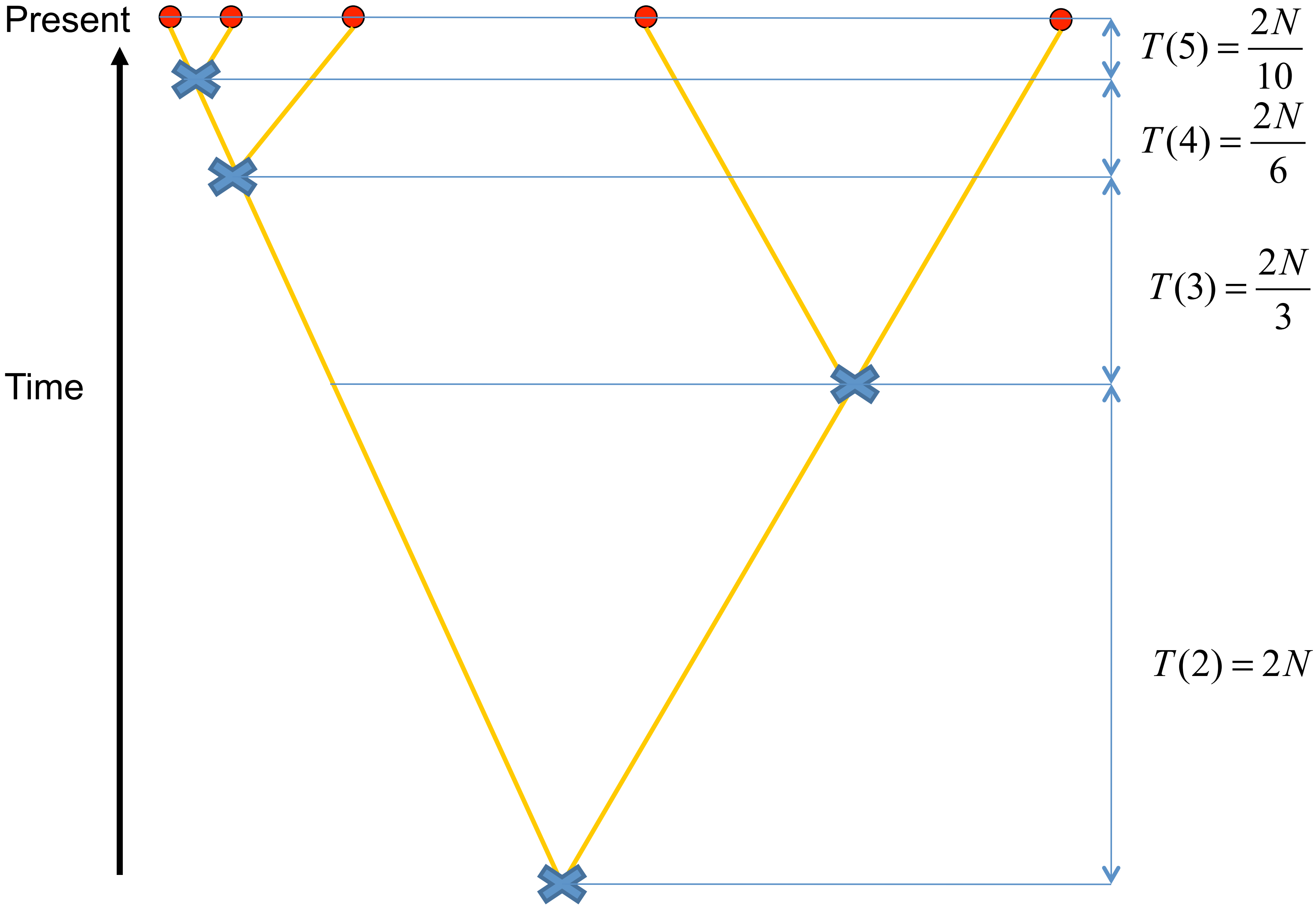
$$P = \frac{k*(k-1)}{4N}$$

Kingman's "n-coalescent"

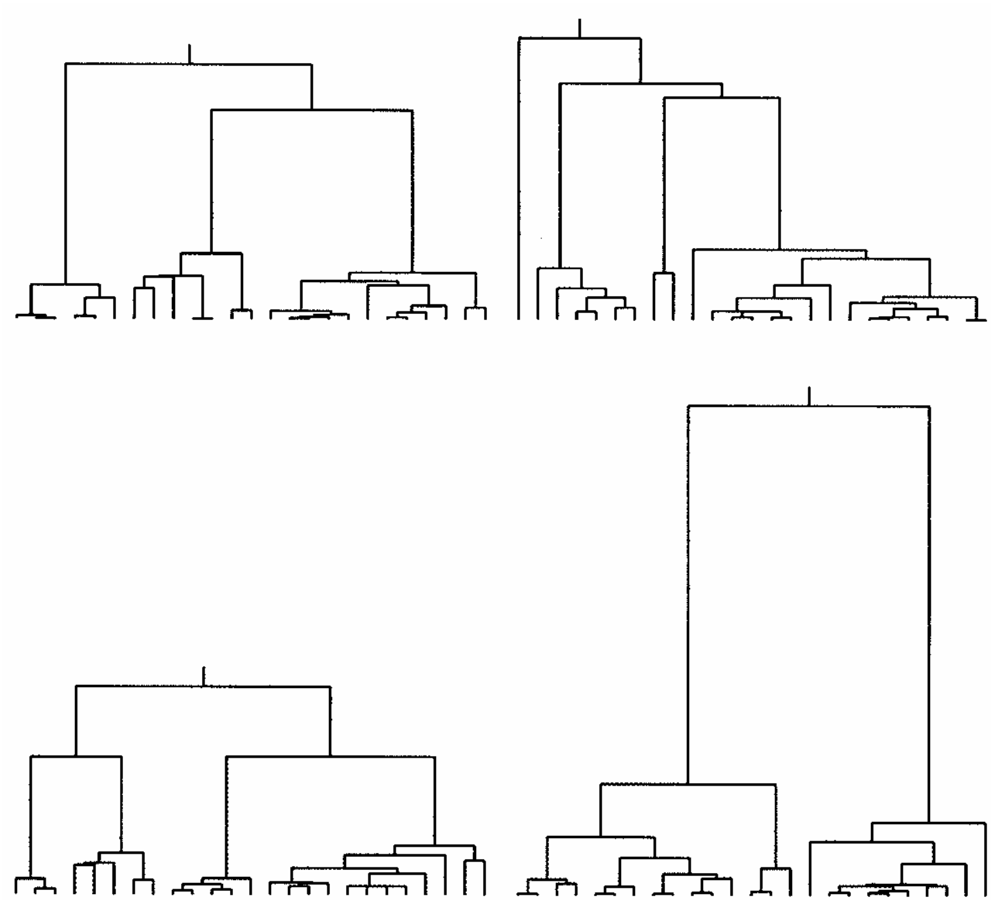
- now depends on k
- The time to coalesce follows a geometric distribution, but with parameter $P = \frac{k*(k-1)}{4N}$
- It now depends on k and we have:

$$T(k) = \frac{4N}{k*(k-1)}$$

- We still have T(2)=2N


Present
Time
$T(5) = \frac{2N}{10}$
$T(4) = \frac{2N}{6}$
$T(3) = \frac{2N}{3}$
$T(2) = 2N$

Coalescent in a stationary population



Gene genealogies are extremely variable in stationary populations, both for the topology and the branch length

Generally we'll have long internal branch length and small external branch length.

$$T(k) = \frac{4N}{k*(k-1)}$$

Time to coalesce:

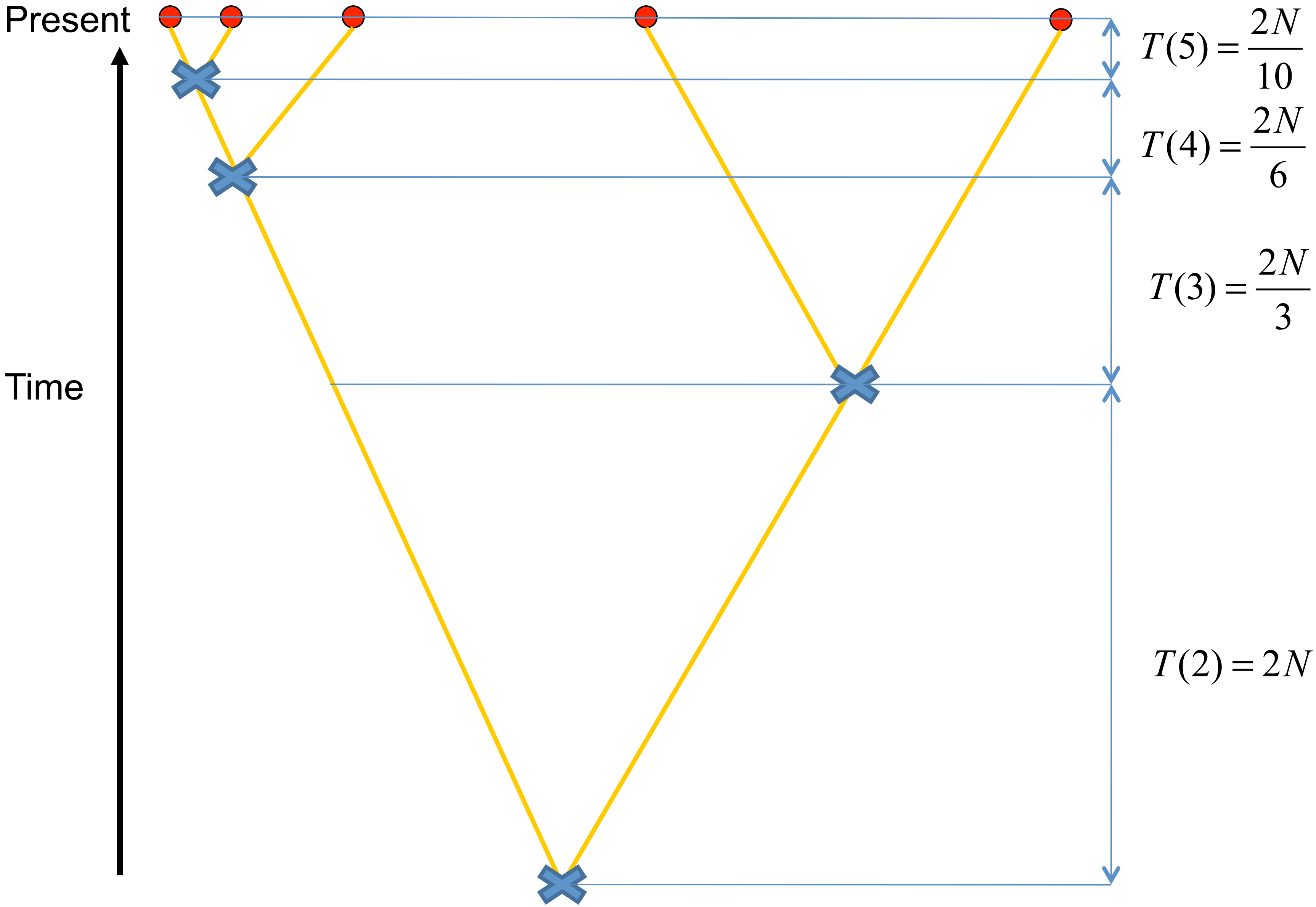
- The total time for the k genes to coalesce is:

$$T_{MRCA}(k) = T(2) + T(3) + \cdots + T(k-1) + T(k)$$

$$= 4N\left(\underbrace{\frac{1}{2*1}}_{=\frac{1}{1}-\frac{1}{2}} + \underbrace{\frac{1}{3*2}}_{=\frac{1}{2}-\frac{1}{3}} + \cdots + \underbrace{\frac{1}{(k-1)*(k-2)}}_{=\frac{1}{k-2}-\frac{1}{k-1}} + \underbrace{\frac{1}{k*(k-1)}}_{=\frac{1}{k-1}-\frac{1}{k}} \right)$$

$$= 4N\left(1 - \frac{1}{k}\right)$$

- The expected time during which there are only two branches (2N) is greater than half the expected total tree height



Present
Time
$T(5) = \frac{2N}{10}$
$T(4) = \frac{2N}{6}$
$T(3) = \frac{2N}{3}$
$T(2) = 2N$

Total length of the tree:

- The total length is simply given by:

$$T_{Total}(k) = 2T(2) + 3T(3) + \cdots + kT(k)$$

$$= 4N\sum_{i=2}^{k}\frac{i}{i(i-1)}$$

$$= 4N\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k-1}\right)$$

$$= 4N\sum_{i=1}^{k-1}\frac{1}{i}$$

$$T(k) = \frac{4N}{k*(k-1)}$$

Sampling effect

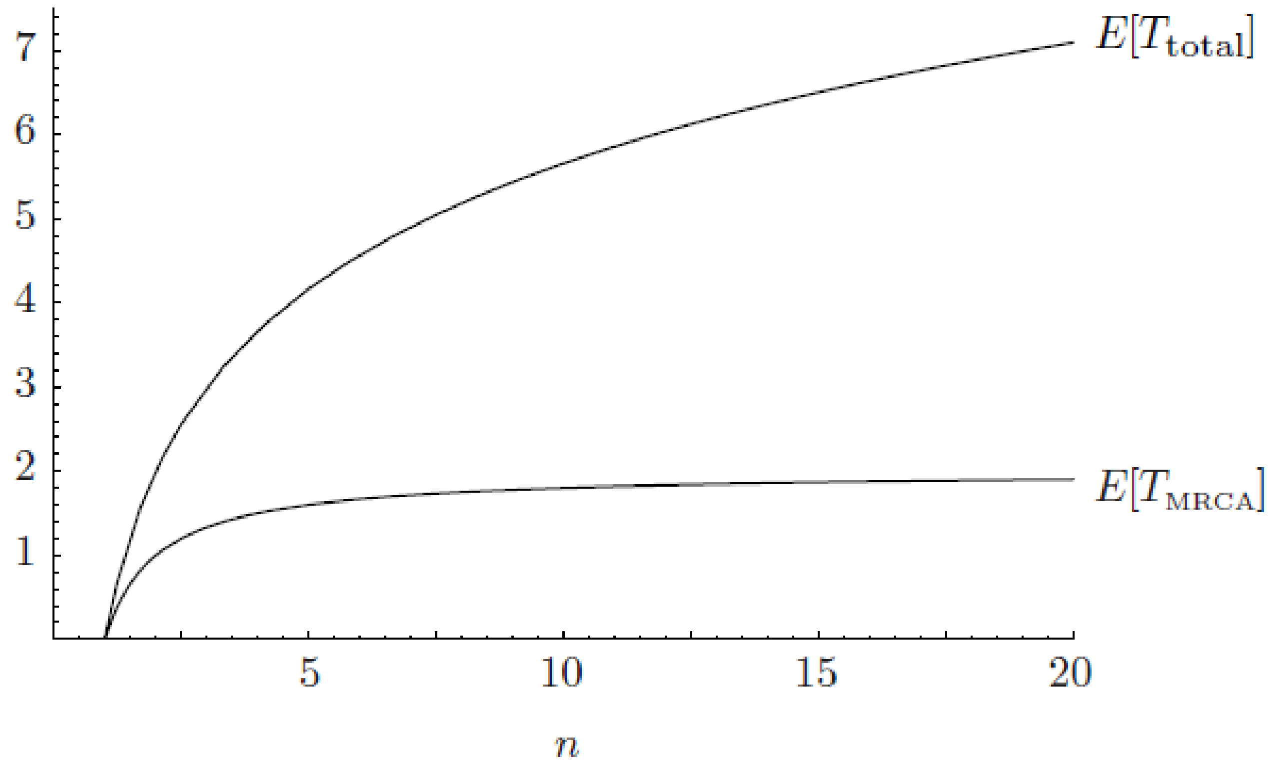


Figure 3.3: The relationship between sample size and the expected values of T_{MRCA} and T_{total}.

Wakeley, 2009

Consequences:

- The larger the sample size the greater the rate of coalescence
- the larger the population size the slower the rate of coalescence
- Time to coalescence gets longer as the process moves toward the most recent common ancestor
- No need to take a lot of genes

Continuous-time version

- The **geometric (discrete)** distribution can be approximated with an **exponential (continuous)** distribution as long as **N is big**:

$$P = \frac{1}{2N}\left(1 - \frac{1}{2N}\right)^{t-1} \approx \frac{1}{2N}\mathrm{e}^{-\frac{t}{2N}}$$

- This is the exponential distribution with parameter $\lambda = \frac{1}{2N}$
- We often also rescale the time so that T=1 corresponds to t=2N: $P = \mathrm{e}^{-T}$

Scaled continuous-time "n-coalescent"

- The probability (density) to have a coalescence event at time T is:

$$P_k(T) = \frac{k(k-1)}{2} e^{-\frac{k(k-1)}{2}T}$$

- This is the basic equation for **genealogies**
- But what can we do with this now?

A first simulation algorithm:

1. Start with k genes
2. Simulate the time T(k) from the exponential distribution with parameter
$$P = \frac{k*(k-1)}{2}$$
3. Choose a random pair of genes and merge them into one
4. Decrease the sample size $k \to k-1$
5. If k>1, go to 2, otherwise stop

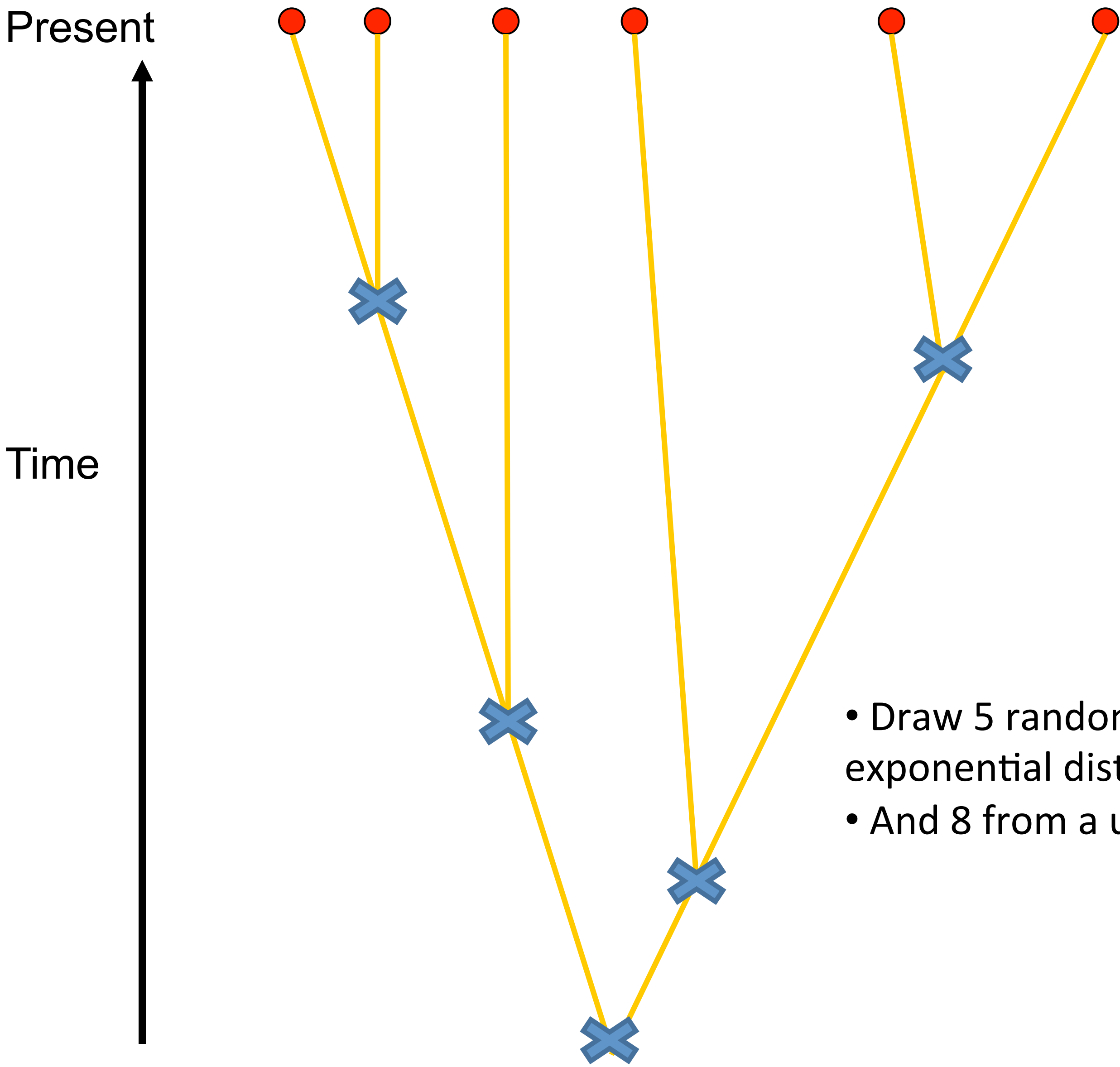
Demo

- ms, fastsimcoal and Figtree

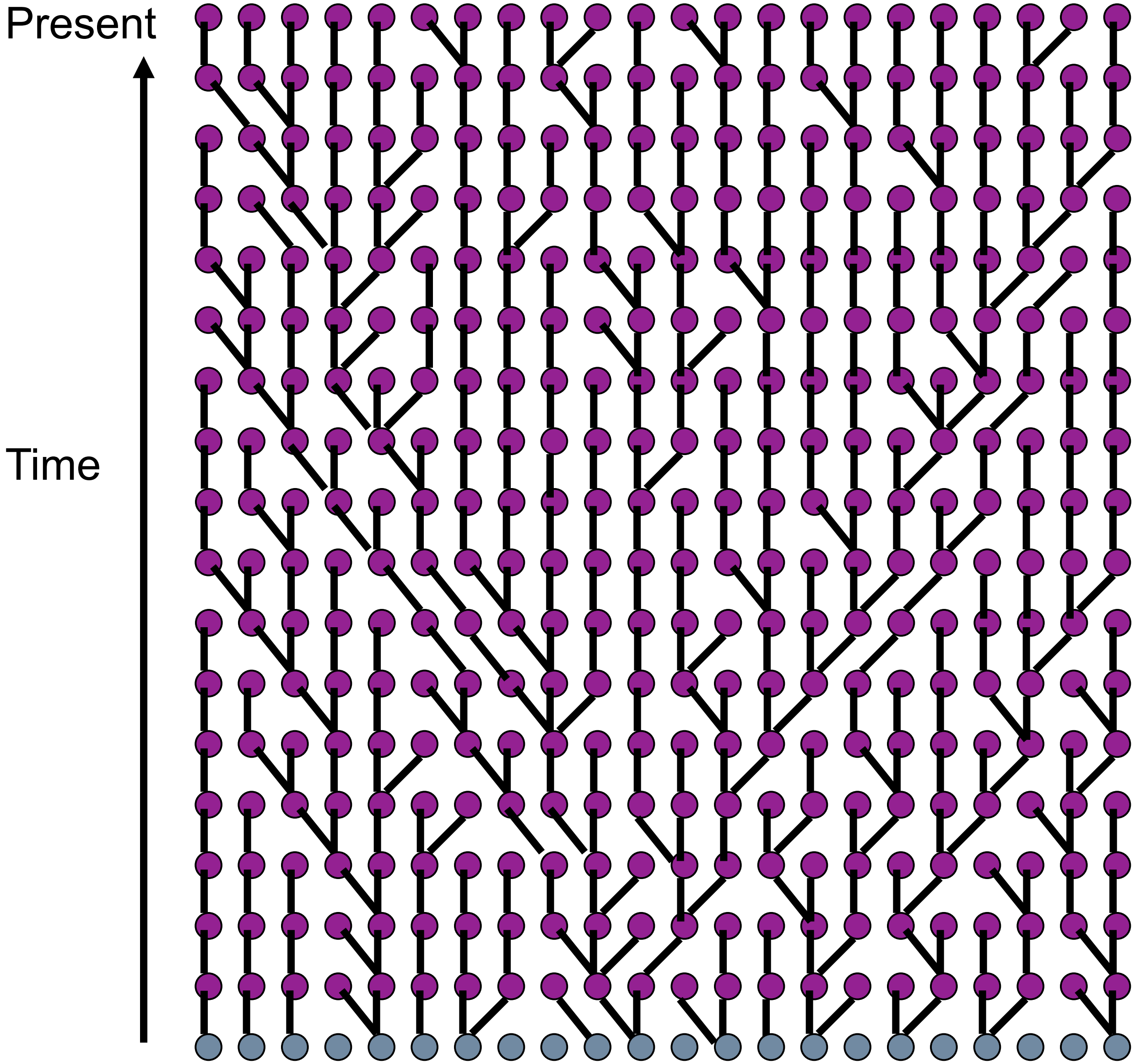
```
./ms 5 1 -T
./fastsimcoal -i 1PopDNA_sta.par —n 10 -T
```

So what?

- This simulation algorithm is extremely **efficient** compared to a forward simulation of the Wright-Fisher model
- You only simulate **what you need**
- The complexity increases **linearly** with the number of genes



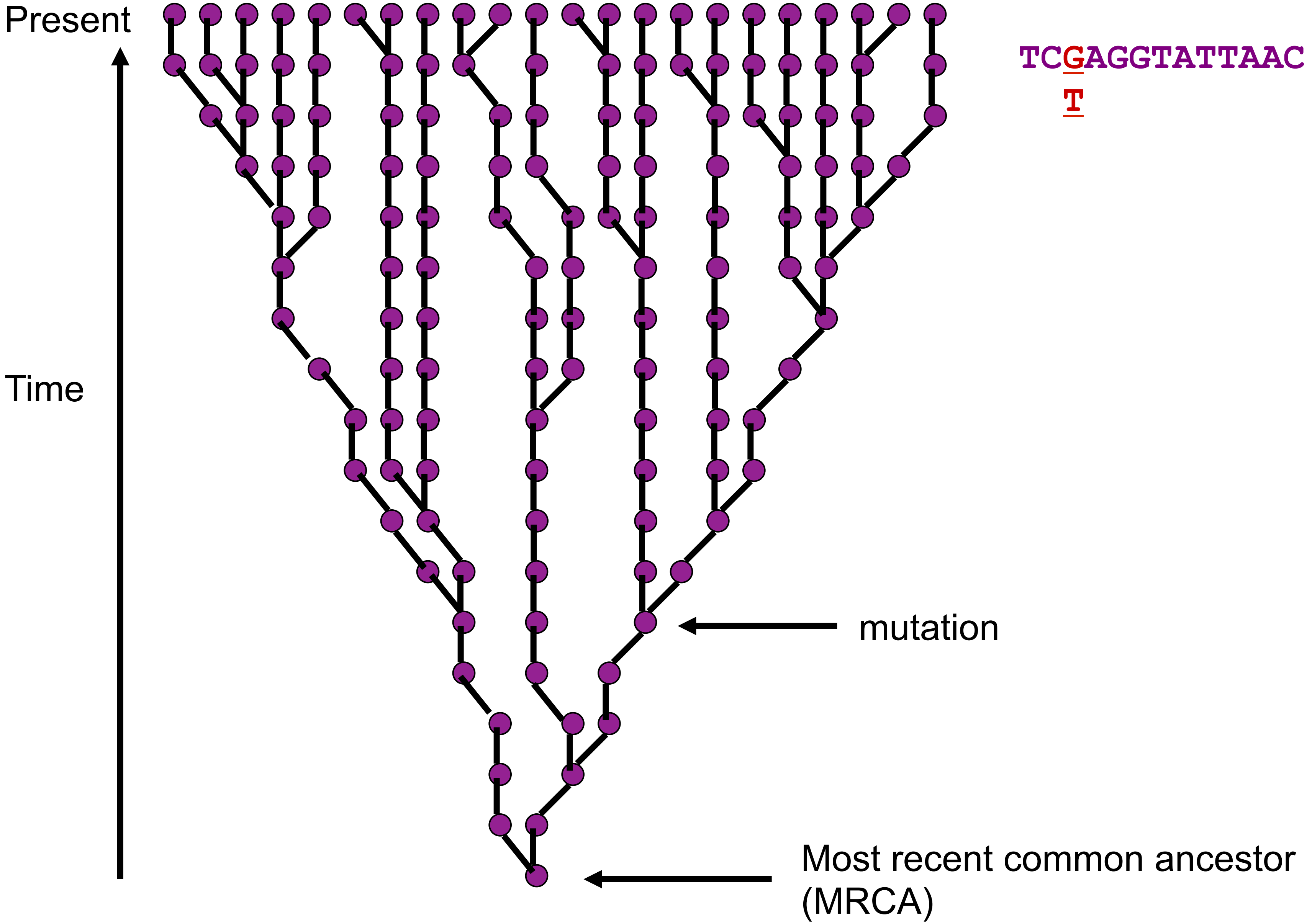
- Draw 5 random numbers from an exponential distribution
- And 8 from a uniform distribution

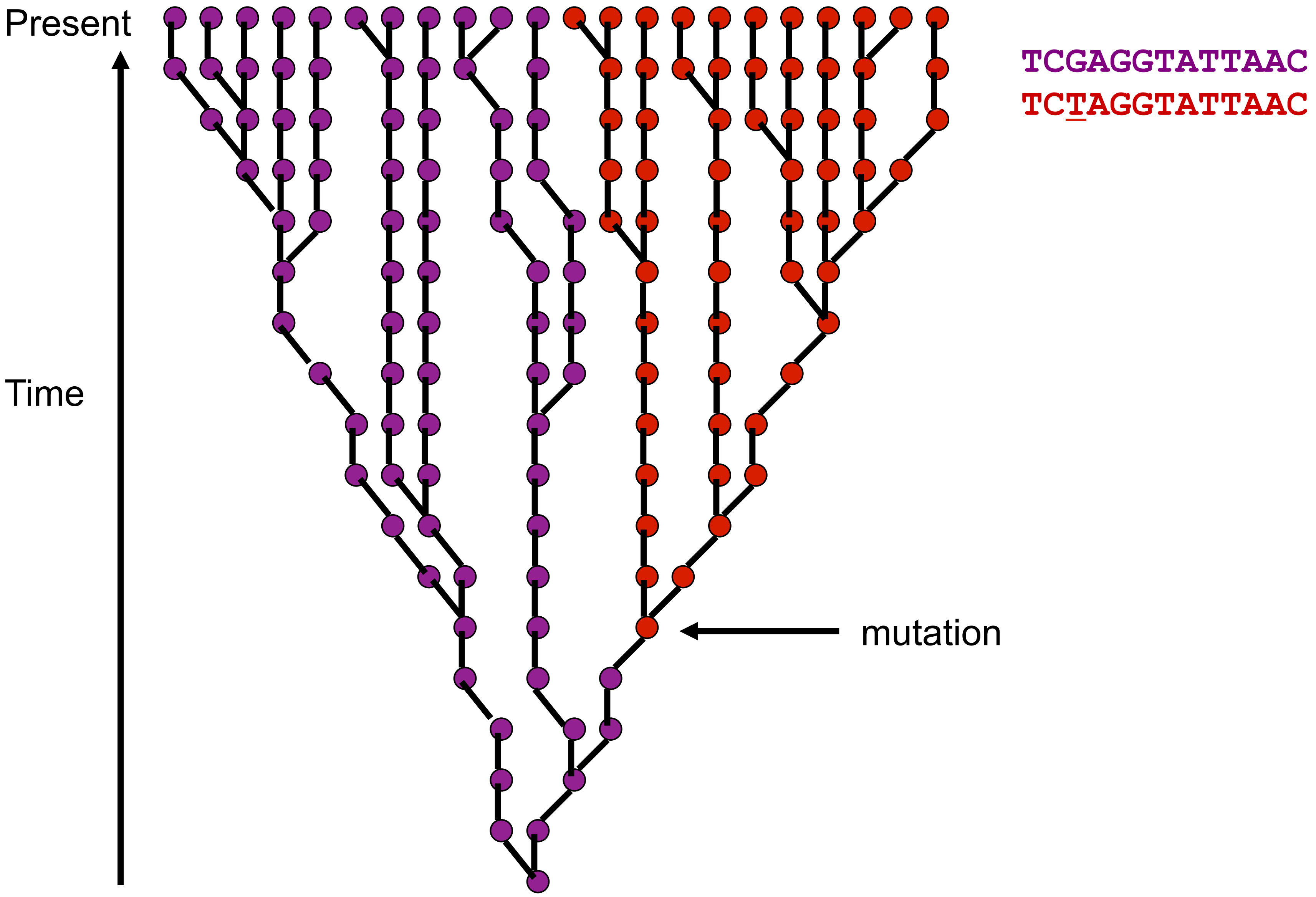


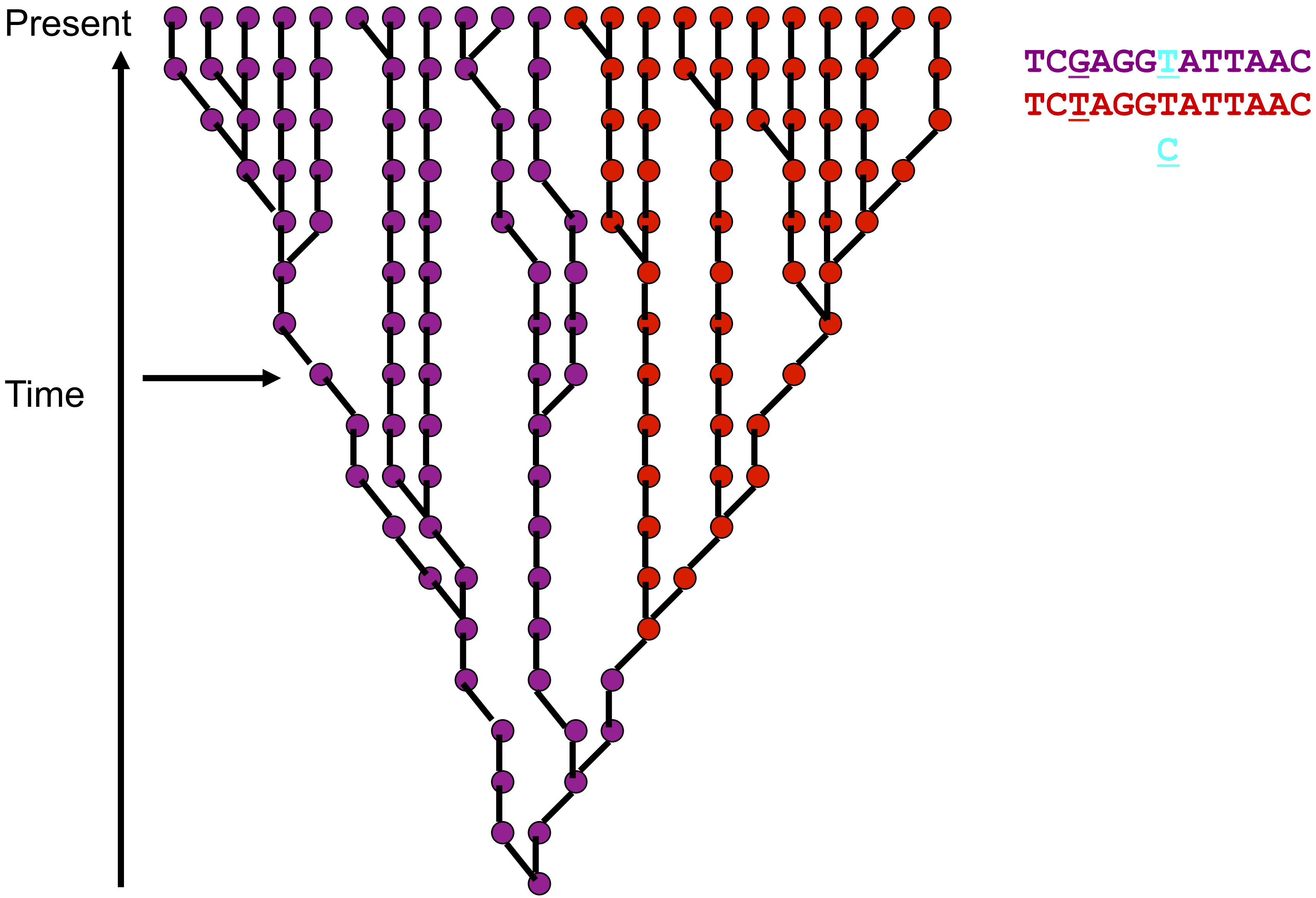
And here???

Adding neutral mutations

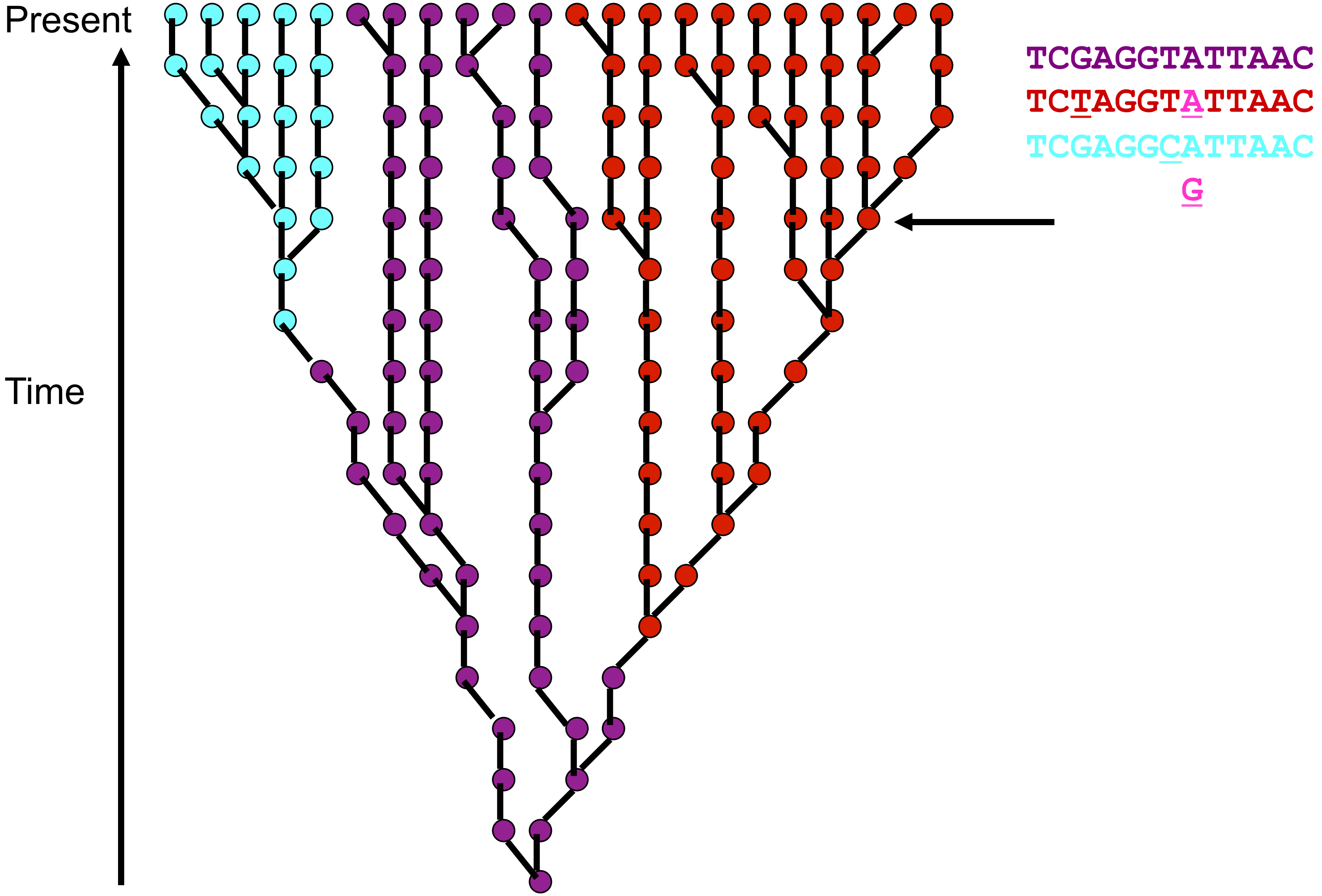
- The shape of neutral coalescent trees only depend on the population demography, and not on the mutational process. The mutational process can be modeled as an independent process superimposed on a realized coalescent tree.

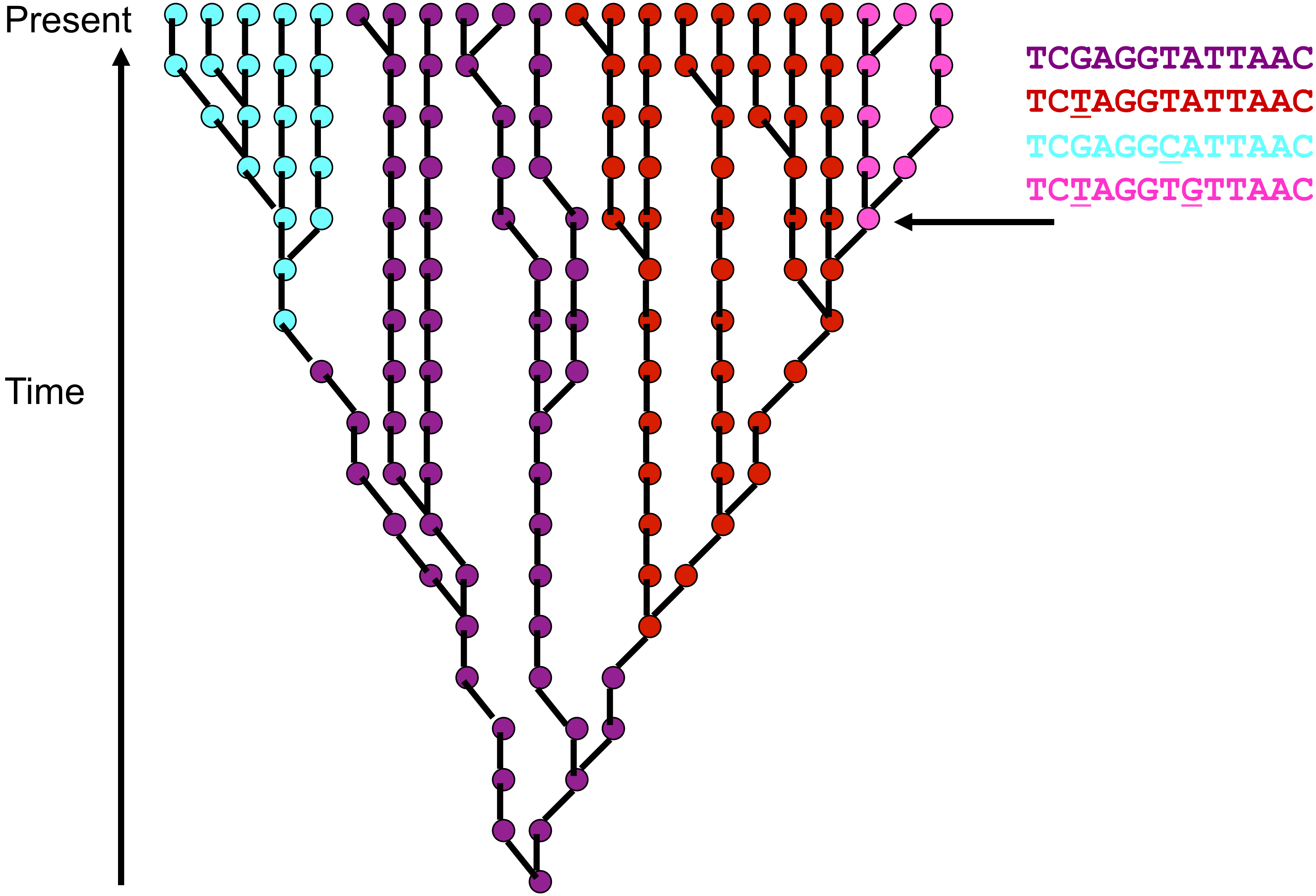

Present
Time
TCGAGGTATTAAC
T
mutation
Most recent common ancestor
(MRCA)


Present
Time
mutation
TCGAGGTATTAAC
TCTAGGTATTAAC

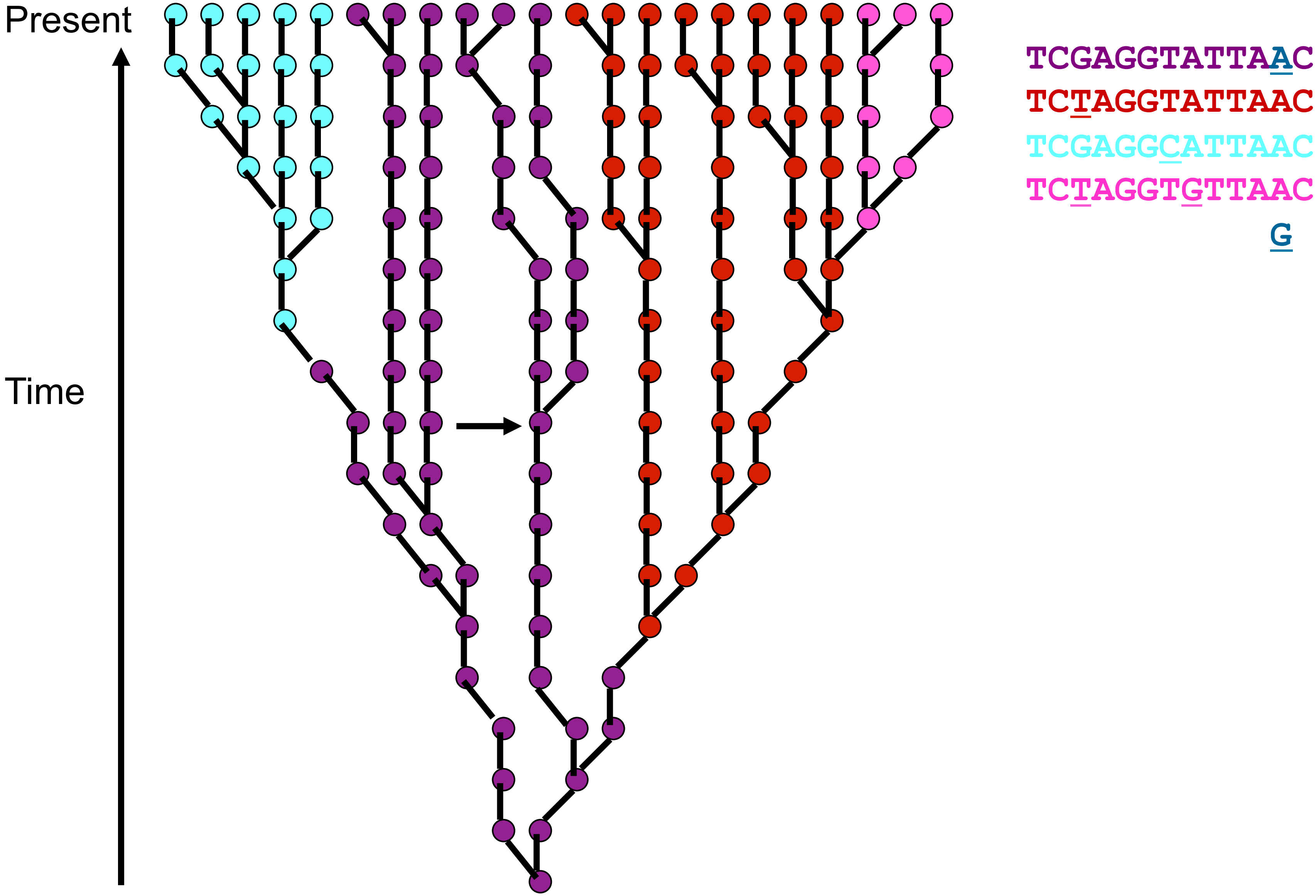

Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
C

Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC


Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC
G

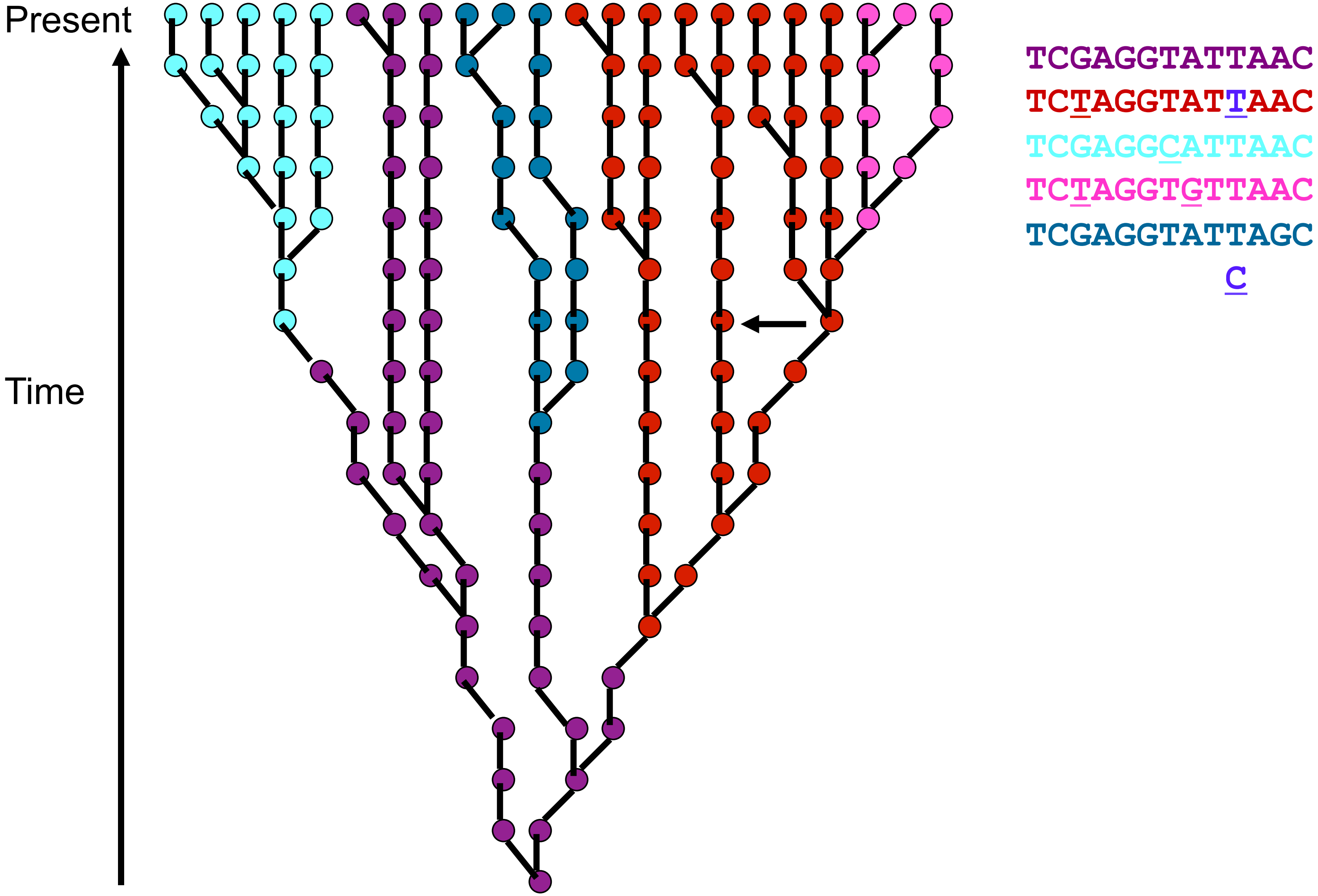


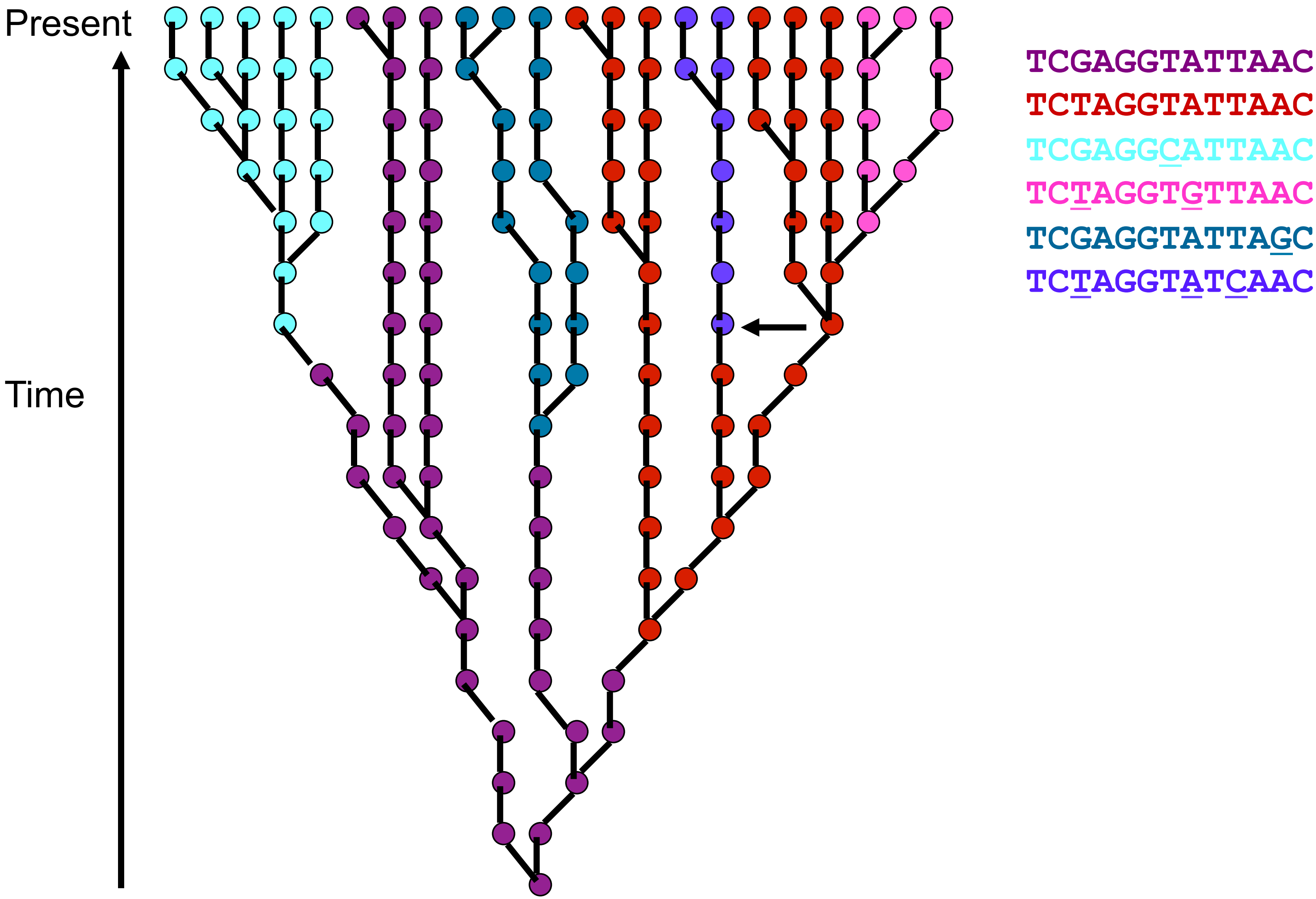
Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC
TCTAGGTGTTAAC

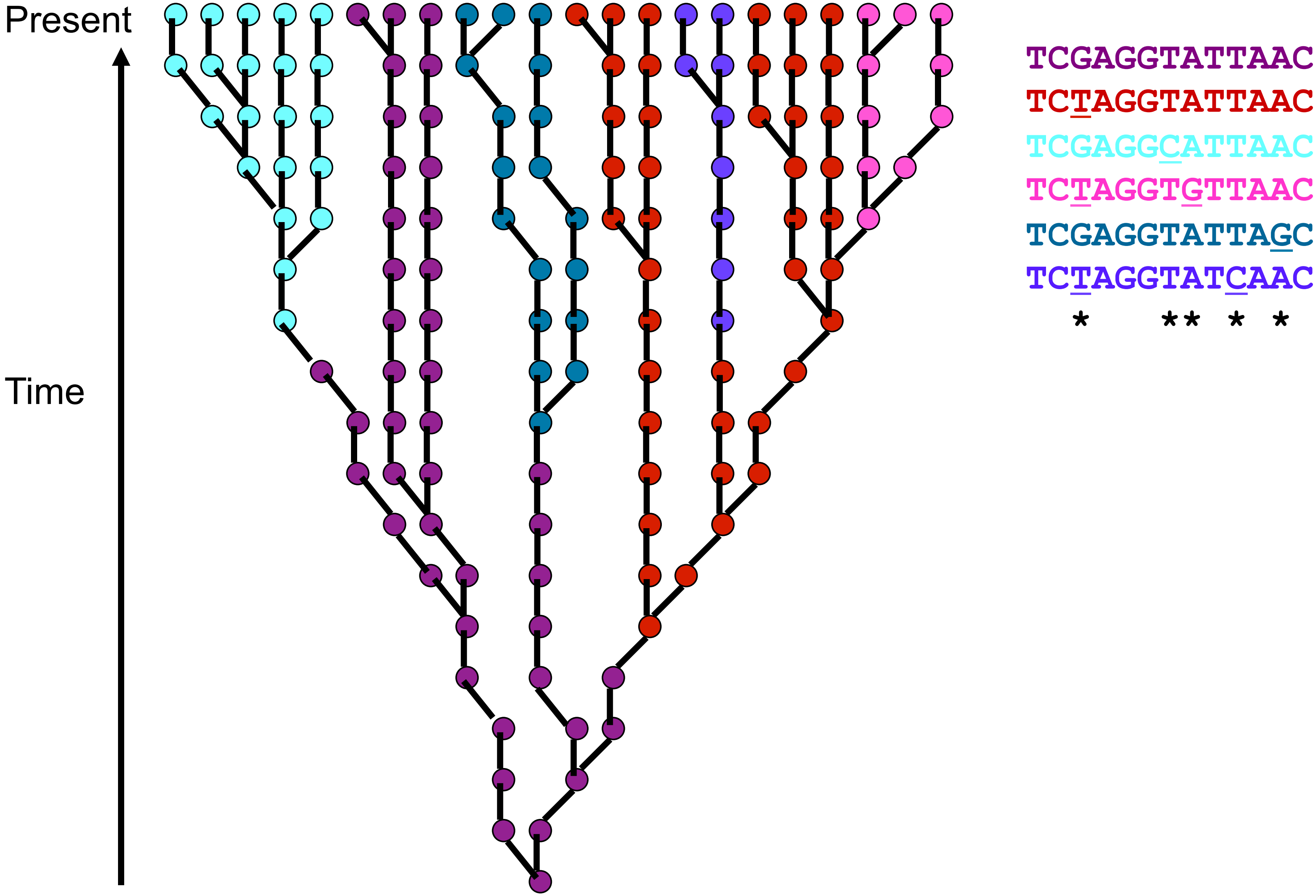


Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC
TCTAGGTGTTAAC
G

Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC
TCTAGGTGTTAAC
TCGAGGTATTAGC


Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC
TCTAGGTGTTAAC
TCGAGGTATTAGC
C


Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC
TCTAGGTGTTAAC
TCGAGGTATTAGC
TCTAGGTATCAAC

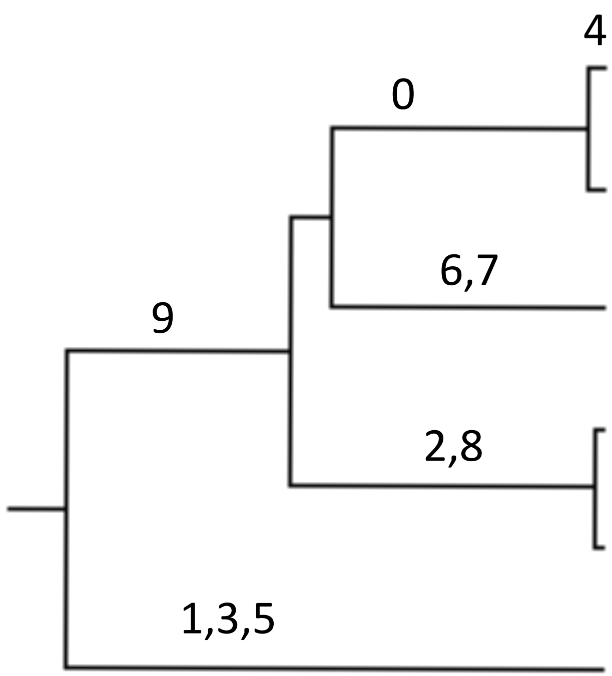

Present
Time
TCGAGGTATTAAC
TCTAGGTATTAAC
TCGAGGCATTAAC
TCTAGGTGTTAAC
TCGAGGTATTAGC
TCTAGGTATCAAC
*   ** * *

Simulating population data

- Mutations just accumulate along the branches of the tree according to a Poisson process with rate $\lambda = \mu t$ for a branch of length t.
- The Poisson process is stochastic but it should be immediately obvious that long branches will carry more mutations than short branches

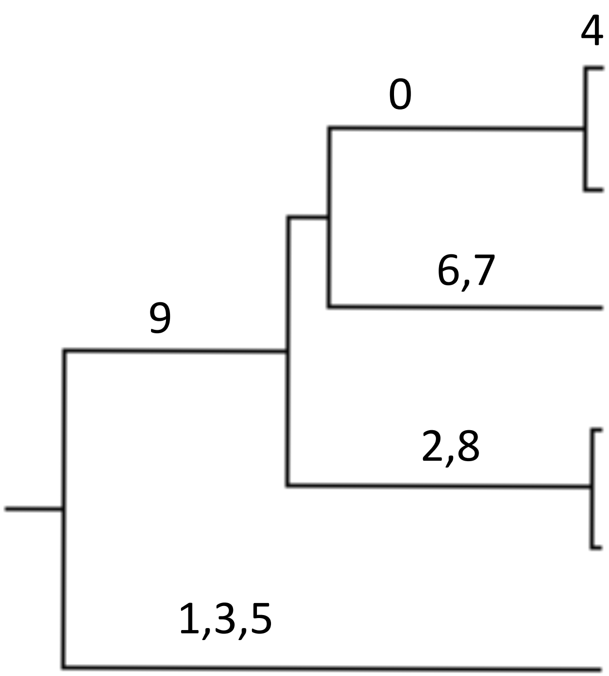
Simulating population data

- Generate a coalescent (Topology + Branch lengths)
- For each branch length t, drop mutations with rate μt
- Based on infinite sites, each mutation is at a unique location



Simulating population data

- Generate Sequences



0	1	2	3	4	5	6	7	8	9
1	0	0	0	1	0	0	0	0	1
1	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	1	1	0	1
0	0	1	0	0	0	0	0	1	1
0	0	1	0	0	0	0	0	1	1
0	1	0	1	0	1	0	0	0	0

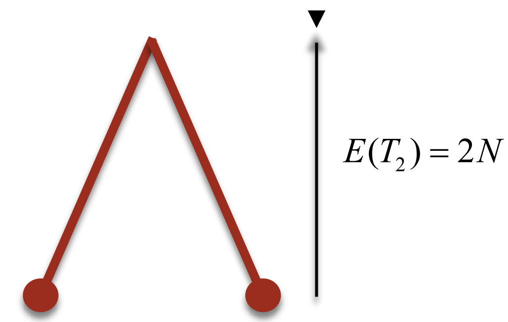
Demo

- ms, fastsimcoal, Figtree, Arlequin input file

```
./ms 6 1 -s 10 -T
./fastsimcoal -i 1PopDNA_sta.par –n 10 -T
```

Average number of pairwise differences (π)

- Since the expected coalescent time between a pair of gene is 2N generations, the average number of mutations expected between a pair of genes (also called the average number of pairwise differences under the infinite site model) is:



$$E(\pi \mid N, \mu) = 2 \times E(T_2 \times \mu) = 2\mu \times 2N = 4N\mu = \theta$$

- This shows that coalescent theory provides a very powerful way to obtain classical population genetics results.

Number of segregating sites (S)

- It is very simple to derive the expected number of segregating (polymorphic) sites S in a sample of size n under the infinite site model as:

$$E(S \mid n, N, \mu) = E(T_{total} \times \mu) = \mu E(T_{total}) = 4N\mu \sum_{i=1}^{n-1} \frac{1}{i} = \theta \sum_{i=1}^{n-1} \frac{1}{i}$$

- A result that was originally obtained by Ewens (1974) and Watterson (1975) using much more complex approaches based on classical forward population genetics.

Demo

- Fastsimcoal, arlsumstat, R

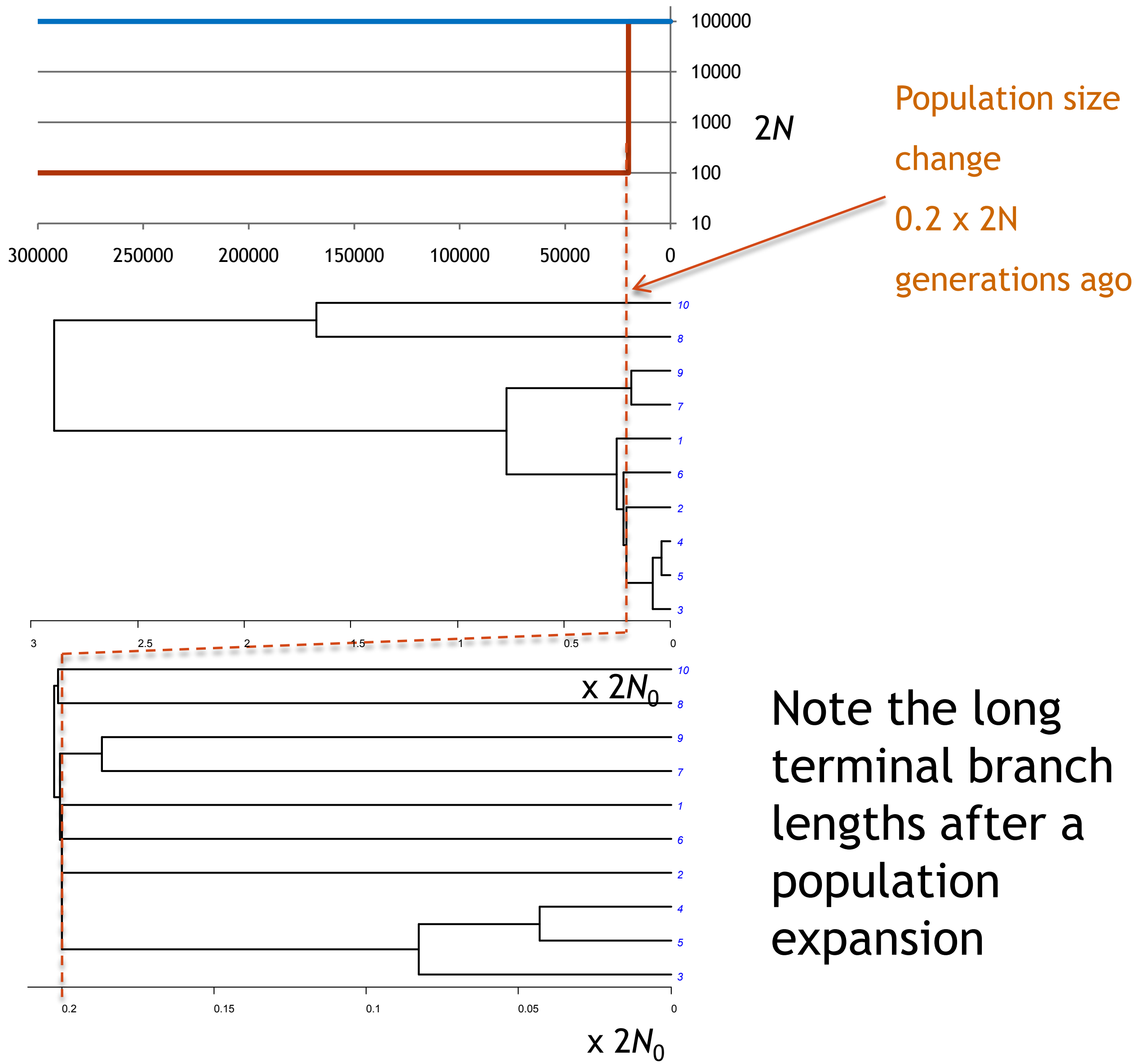
```
./fastsimcoal -i 1PopDNA_sta.par -n 100
./LaunchArlsumstatDirMac.sh 1PopDNA_sta  SettingsDNAStats.ars stats.txt

stats=read.table("1PopDNA_sta/stats.txt",header=T)

hist(stats$Pi_1)
theta=2*20000*0.00000002*100000
mean(stats$Pi_1)

hist(stats$S_1)
theta*sum(1/(1:9))
mean(stats$S_1,br=20)
```

Variable population size

- Intuitively, coalescent events will tend to be rare when the population size is large and frequent when it is small.
- Actually population size changes only require a rescaling of branch lengths and have no effect on the topology of the tree.
- Assuming that current population size is N_0 and that t generations ago it was $N(t) = N_0\lambda(t)$, then a branch generated under a coalescent process occurring at rate N_0 between times t_1 and t_2 should just be rescaled by a factor:

$$\Lambda = \int_{t_1}^{t_2} \frac{1}{\lambda(s)} ds$$

Past demographic expansion

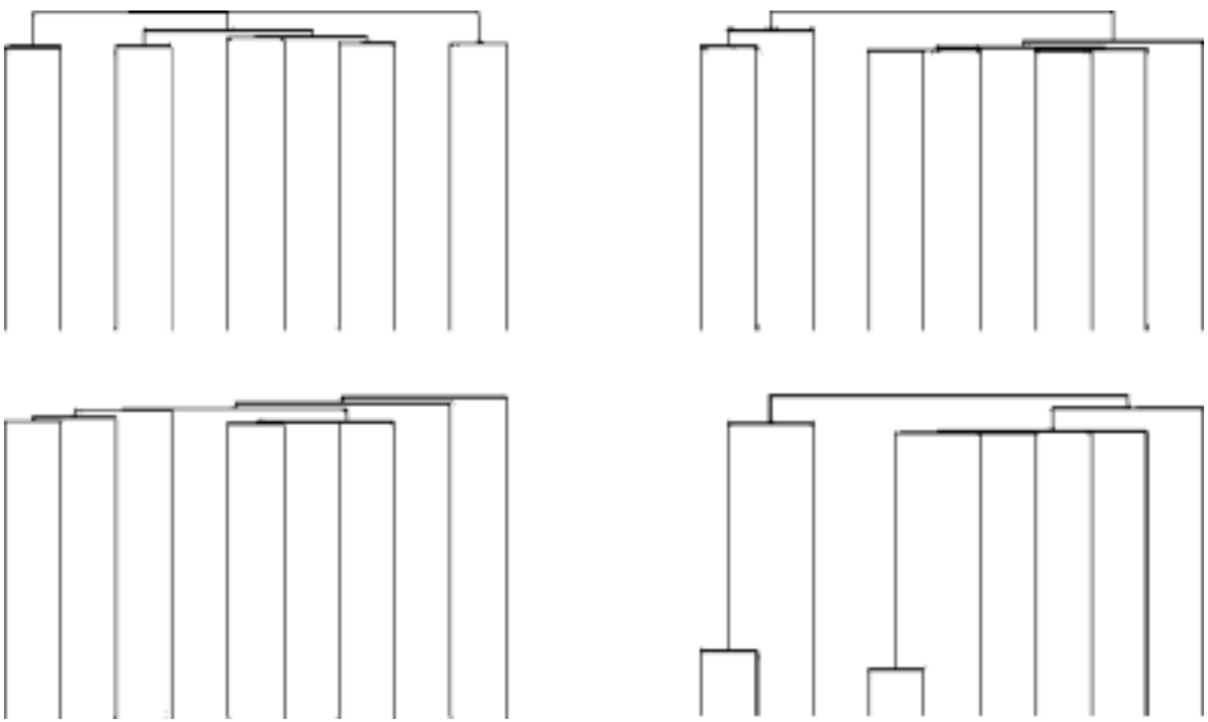


Coalescent in a stationary population

Rescaled coalescent

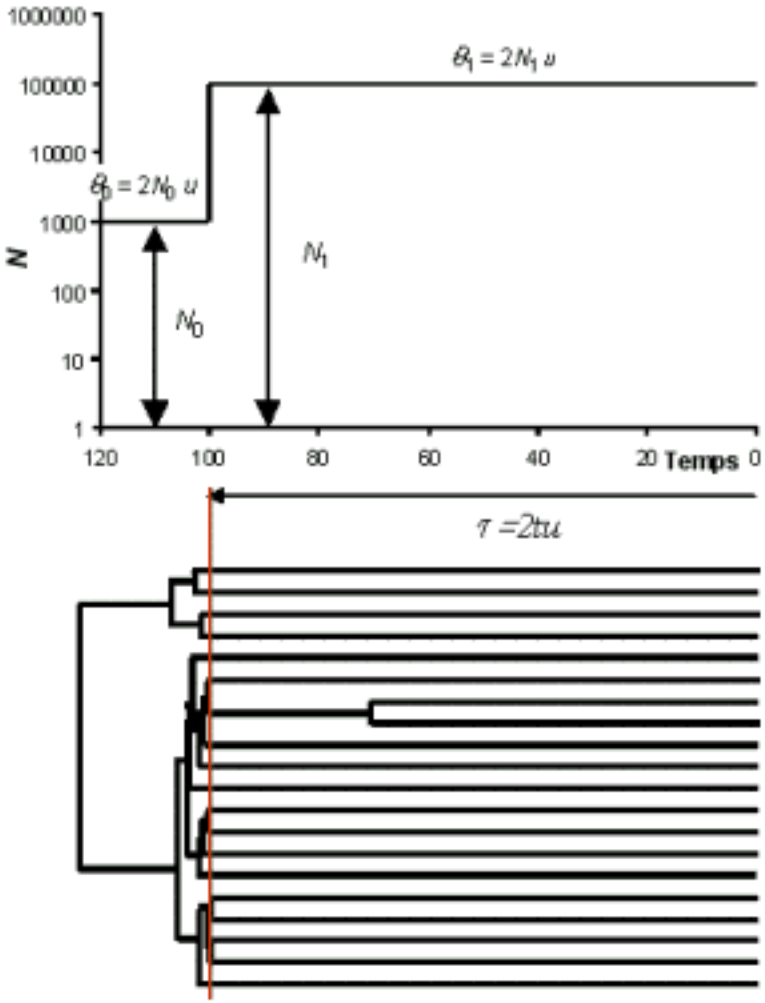
Note the long terminal branch lengths after a population expansion

Past demographic expansion

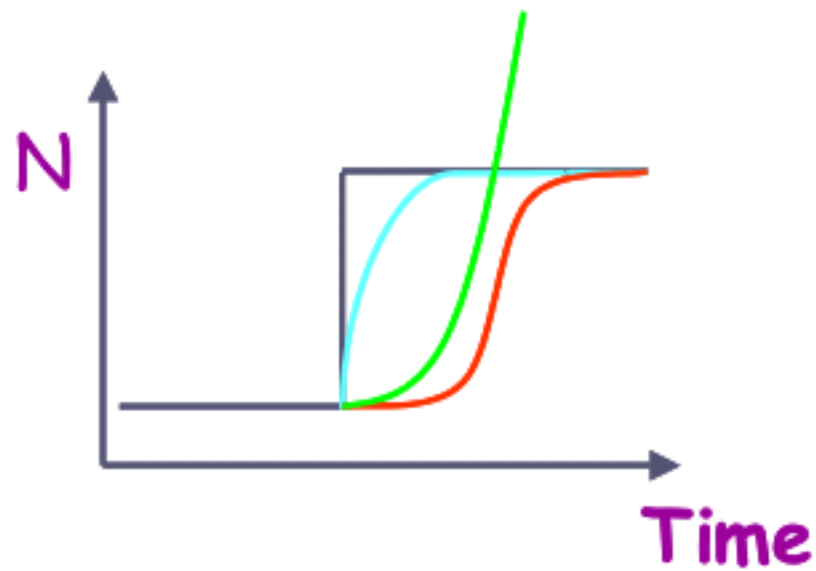


Genealogies in expanding populations have usually short internal branch lengths and long external branch lengths. Comb-like or star-like genealogies

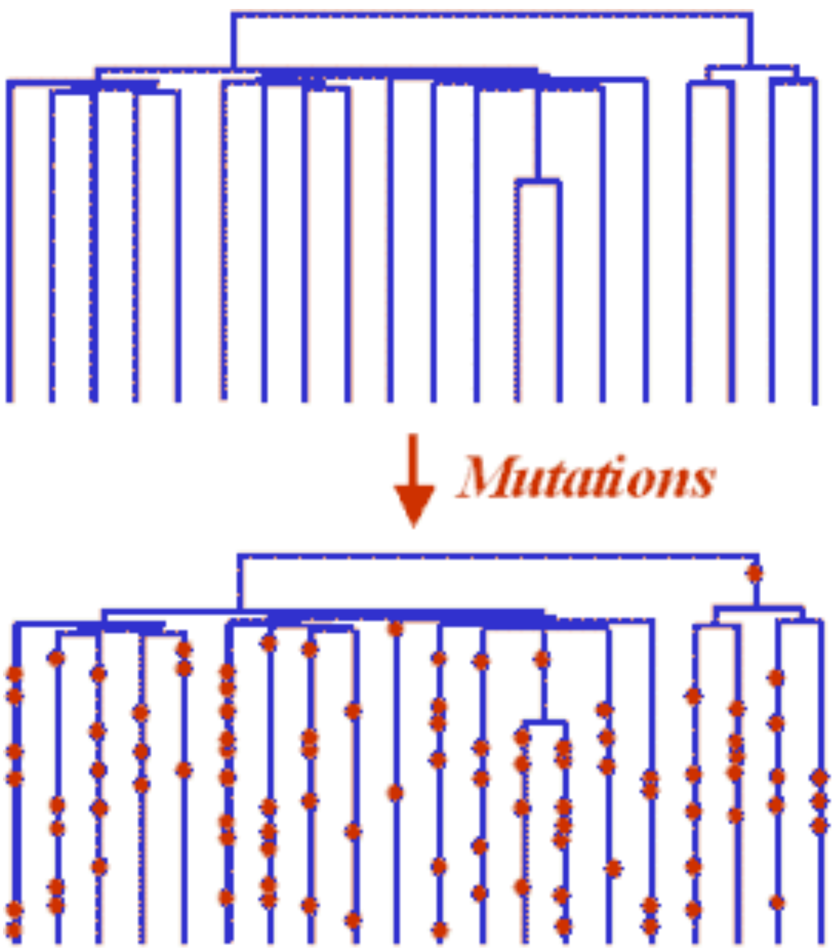
The effect of a sudden expansion



Coalescent events are very unlikely in large populations, but much more likely in small populations

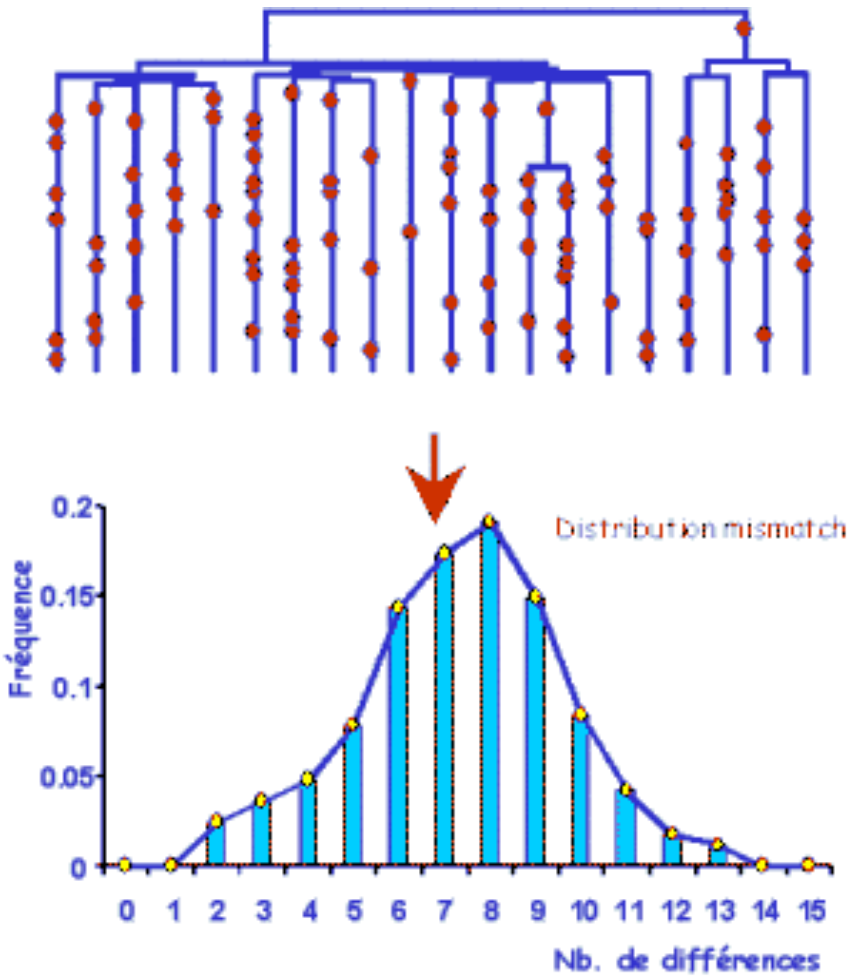


Mutations in expanding populations



Mutations will accumulate preferentially after the expansion

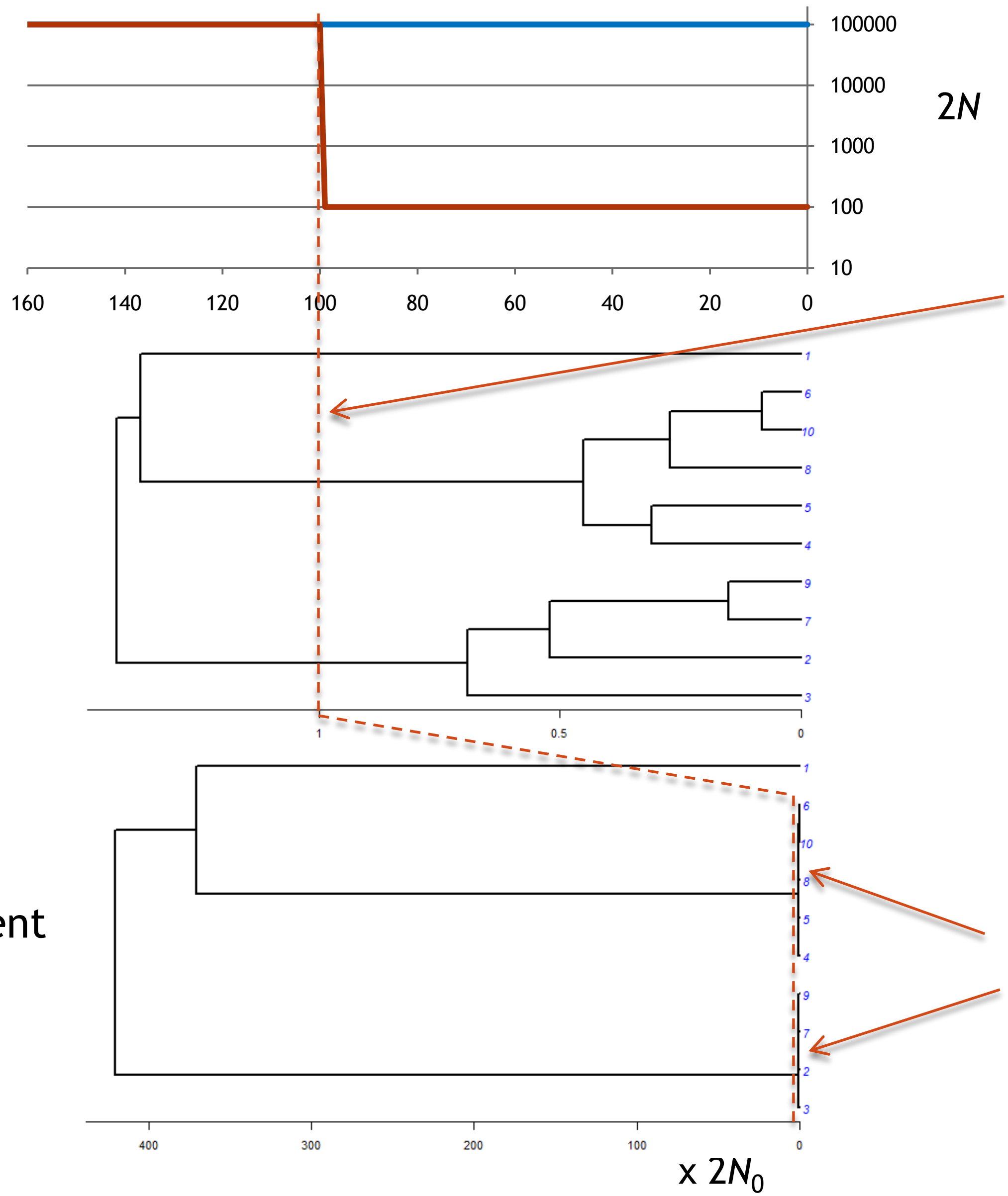
Mismatch distribution



The molecular diversity of a sample may be summarized by plotting the distribution of the number of pairwise differences between genes

This distribution is often called the **mismatch distribution**

Past demographic contraction

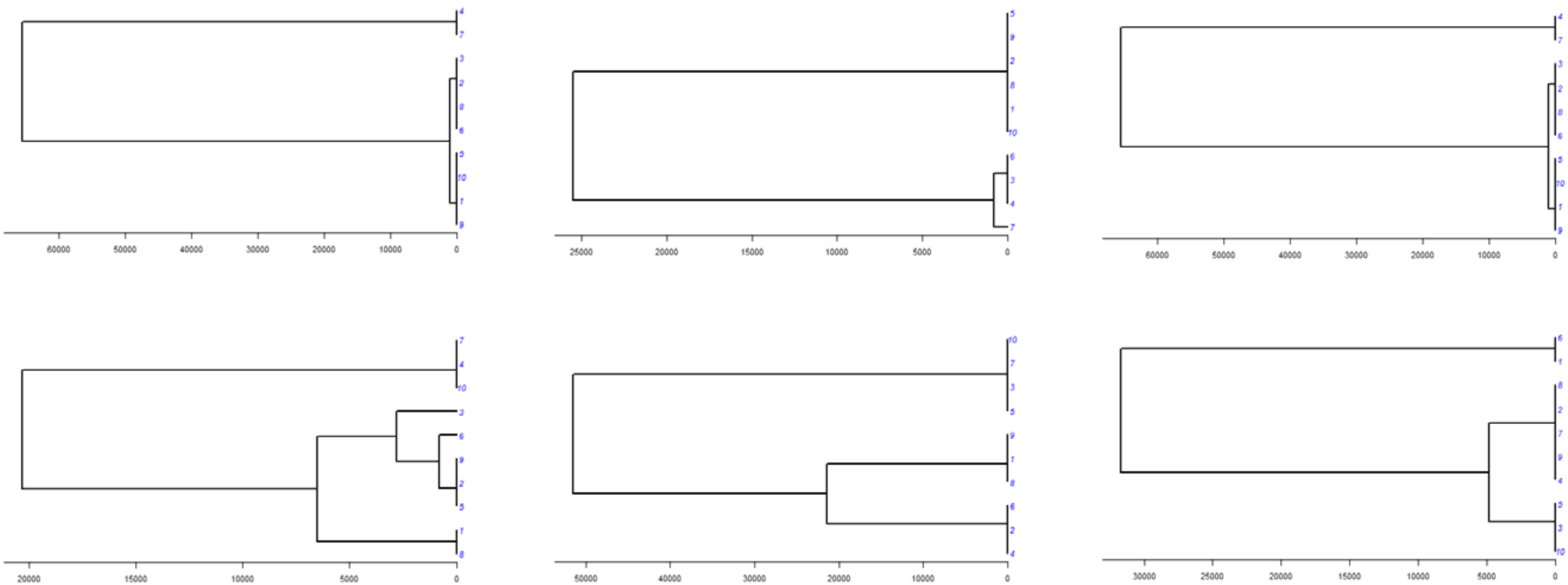


Coalescent in a stationary population

Rescaled coalescent

Note the long internal branches and tiny external branches after a population contraction

Past demographic contraction



Contractions often leads to the observation of deep lineages and two main clades

Demo

- Fastsimcoal, Figtree, arlsumstat, R

```
./fastsimcoal -i 1PopDNA_bot.par -n 100 -T

./fastsimcoal -i 1PopDNA_exp.par -n 100 -T
./LaunchArlsumstatDirMac.sh 1PopDNA_exp  SettingsDNAStats.ars stats.txt

stats=read.table("1PopDNA_exp/stats.txt",header=T)

hist(stats$Pi_1)
```

Simulating the coalescent

- A big advantage of coalescent approaches is that they lead themselves to very efficient simulations, as compared to forward approaches.
- Advantages:
  - Speed
  - Small memory footprint
  - Direct simulation of the sample, no sub-sampling
  - No need to specify initial conditions (initial allele frequencies)
  - Easy to integrate into estimation procedures (ABC, likelihood-based...)
- Disadvantages:
  - Approximation (multiple coalescent events not allowed)
  - Difficult to simulate non-neutral diversity
  - Difficult to include realistic factors linked to life-history traits
  - Simulations involving recombination become tedious to program