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Neuronetworks February 2014
Synchronisation

Synchronisation in Neural Networks

In general synchronisation is not at all easy to understand mathematically! The standard model nowadays is the Kuramoto model [1] and an account on other approaches can be found in [2] (links to articles can be found at the course web-page).

However a simple two-neuron case can be analysed fairly easily. The following approach is a simplification of the Kuramoto model.

Suppose we have two neurons A and B which fire with a periodic frequency. The interval between firings of A and B are both 1 if no other input is given. To be more precise: whenever A fires, it is expected to fire again after an interval of 1 unless it receives an input from B and similarly for B. Suppose initially they are not connected and A fires at times $a_0, a_1, \ldots$ and B fires at times $b_0, b_1, \ldots$ and we have $a_{n+1} - a_n = b_{n+1} - b = 1$ for all n.

If B receives a signal from A, it gets excited and the expected time to lapse before it fires again changes. Suppose $f\colon [0,1] \to [0,1]$ is the function that tells us how much it changes. Thus if at time t, B is expected to fire in time u and A fires, then the new expected time before B fires is $f(u)$. Symmetrically for A. If the neurons are excitatory, then a natural assumption is $f(u) < u$, but the following analysis can be carried out independently of that.

Let us look at the dynamics when the neurons are connected. Suppose that first A fires at time a_0 and then B at time b_0. Denote the length $b_0 - a_0$ by u. Now if B didn't fire, A would fire at point $a_1 = a_0 + 1 = b_0 + (1-u)$. But since B fired, this changes to $b_0 + f(1-u)$. After b_0, B would normally fire at $b_1 = b_0 + 1 = b_0 + f(1-u) + (1 - f(1-u))$, but since A fires at $b_0 + f(1-u)$, this changes to $b_0 + f(1-u) + f(1 - f(1-u))$. This analysis shows that if the time from the previous time A fired to the time B fired was u, then the time from the next time A fires to the next time B fires is $g(u) = f(1 - f(1-u))$. So if we denote by $\bar{a}_n$ and $\bar{b}_n$ the times when the connected neurons fire and assume $\bar{a}_0 < \bar{b}_0$, then for all n we have $b_{n+1} - a_{n+1} = g(b_n - a_n)$.

Obviously we have to look now at the sequence

$$g(u), g(g(u)), g(g(g(u))), \ldots, g^n(u), \ldots,$$

and if it converges either to 0 or to 1, then the neurons synchronise.

In the Kuramoto model, $f(u)$ is usually $u - sin(u)$ and the interval is assumed to be $[0, 2\pi]$ instead of $[0,1]$. This is motivated by the idea that if A is just a little bit ahead of B then it slows down and if it is a little behind, then it tries to catch up. In this case $g(u) = f(f(u)) = f^2(u)$ and so the limit of $(g^n(u))_n$

is the same as the limit of $(f^n(u))_n$ if it exists. Now if $0 < u < \pi$, then $f(u)$ is a convex function with $f(0) = 0$, so $f^n(u) \to 0$. If $\pi < u < 2\pi$, symmetrically $f^n(u) \to 1$.

Getting back to the interval $[0, 1]$, if f is any function such that there is $u_* \in [0, 1]$ such that $g(u)$ is convex for $0 < u < u_*$ and concave for $u_* < u < 1$, then synchronisation takes place.

It is not difficult to see that if f is convex (e.g. second derivative is positive), $f(0) = 0$ and $f(1) = 1$, then this holds if $f(u_0) < 1 - u_0$ where u_0 is the point at which $f'(u_0) = 1$. (Exercise)

References

[1] Juan A. Acebrn, L. L. Bonilla, Conrad J. Prez Vicente, Flix Ritort, and Renato Spigler: *The Kuramoto model: A simple paradigm for synchronization phenomena* Rev. Mod. Phys. 77, 137 – Published 7 April 2005

[2] Anja K. Sturm, Peter Knig: *Mechanisms to synchronize neuronal activity*, Bio. Cybern. 84, 153 – 172 (2001)