Department of mathematics and statistics

Topology I

Final exam 14.11.2013

You may keep with you a one-sided size A4 "memory-helper" during the exam.

1. Let e be the discrete $\{0,1\}$-metric on the real line $\mathbb{R}$. Show that

$$d(x,y) = |x_1 - y_1| + e(x_2, y_2), \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2,$$

defines a metric d in the plane $\mathbb{R}^2$. Determine the open ball $B(\overline{0}, 1)$ in this metric, where $\overline{0} = (0,0)$.

2. Let (X, d) be a metric space and $A \subset X$ a subset. Define the set $int(A)$ consisting of *inner points* of A and its *boundary set* ∂A. Show that A is an open set if and only if

$$A \cap \partial A = \emptyset.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let

$$G(f) = \{(x, f(x)) : x \in \mathbb{R}\}$$

be its graph. Show that the metric spaces $\mathbb{R}$ and $G(f)$ are homeomorphic. The real line $\mathbb{R}$ is equipped with its usual metric and $G(f)$ has the metric induced by the euclidean metric of the plane $\mathbb{R}^2$.

4. Let $A \subset \mathbb{R}^2$ be a closed and bounded set. Show that there is a point $(a, b) \in A$, such that

$$a + b \leq x + y$$

for all $(x, y) \in A$. *Hint:* continuous functions on compact sets.

5. (i) Define a *connected* metric space (X, d).

(ii) Is the metric space $(\mathbb{R}^2, e)$ connected, where e is the discrete $\{0,1\}$-metric in the plane? Please explain your argument.