Stochastic analysis, spring 2013, Final Exam

1. Let $(B_t^{(1)}, \dots, B_t^{(n)} : t \geq 0)$ continuous local martingales in the filtration $\mathbb{F}$ with

$$\langle B^{(i)}, B^{(i)} \rangle_t = t,$$
$$\langle B^{(i)}, B^{(j)} \rangle_t = E_P\big(B_t^{(i)} B_t^{(j)}\big) = c_{ij} t, \text{ for } i \neq j, .$$

with $c_{ij} \in [-1, 1]$ constant.

(a) Each $B_t^{(i)}$ is a Brownian motion. Why ?

(b) Assume $B_0^{(i)} = 0$ at time $t = 0$.
Use inducively Ito formula and Fubini Theorem to compute the joint moment at time t:

$$E_P\big(B_t^{(1)} \dots B_t^{(n)}\big) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ t^{n/2} \sum\limits_{\text{pairings}} \prod\limits_{pairs\{i,j\}} c_{ij} & \text{if } n \text{ is even} \end{cases}$$

where when n is even, the sum is over all pairings of $1, \dots, n$ into $n/2$ pairs, where the pairs are disjoint and the elements of the pairs are distince. For each pairing we then take the product over the pairs of the pairing.

Hint: Compute the semimartingale decomposition of the product $B_t^{(1)} \dots B_t^{(n)}$, and show that the local martingale is a true martingale (which therefore has zero expectation).

This is Wick's formula (in the literature usually the proof is based on the moment generating function).

2. (a) Show that an essentially bounded local martingale (that is for some $K < \infty$, $P(|M_t| < K) = 1 \ \forall t > 0$). is a true martingale.

(b) Let B_t a Brownian motion in the filtration $\mathbb{F}$, and
$Z_t = \exp\big(M_t - t/2\big)$.
Show that Z_t is a continuous martingale which is not uniformly integrable.

3. Let (B_t) be a standard Brownian motion, denote $i = \sqrt{-1}$ as usual. Recall that

$$Z(t, \theta) = \exp\left(i\theta B_t + \frac{1}{2}\theta^2 t \right) =$$
$$\cos(\theta B_t) \exp(\theta^2 t/2) + i \sin(\theta B_t) \exp(\theta^2 t/2) = M_t(\theta) + i N_t(\theta)$$

is a complex valued martingale $\forall \theta \in \mathbb{R}$, that is both real and imaginary parts are martingales.

Compute the brackets $\langle M(\theta), M(\theta) \rangle_t, \langle N(\varphi), N(\varphi) \rangle_t, \langle M(\theta), N(\varphi) \rangle_t$.

4. In the setting of exercise 2,

   Compute the Ito-Clarck martingale representation of the square integrable random variable

$$X_T = \sin(\theta B_T)\cos(\varphi B_T) = E\big(\sin(\theta B_T)\cos(\varphi B_T)\big) + \int_0^T Y_s dB_s$$

   i.e. compute the expectation and find the adapted integrand process Y_s.

   Hint. rewrite

$$X_T = cM_T(\theta)N_T(\varphi)$$

   with $c = \exp(-(\theta^2 + \varphi^2)T/2)$, and use integration by parts, to find the martingale decomposition of the product $(M_t(\theta)N_t(\varphi))$.

5. Let $X_T = \exp(\theta B_T)B_T^2$, where $\theta \in \mathbb{R}$.

   a) Show that $X_T \in L^2(\Omega)$.

   b) Compute $E(X_T)$.

   c) Compute the Ito-Clarck martingale representation of X_T. Hint: use Ito formula and integration by parts.

6. (a) Solve the following Ito SDE

$$a) \quad X_t = x + \int_0^t \sqrt{1 - X_s^2} dB_s - \frac{1}{2}\int_0^t X_s ds$$
$$b) \quad X_t = x + \int_0^t \sqrt{1 + X_s^2} dB_s + \frac{1}{2}\int_0^t X_s ds$$
$$c) \quad X_t = x + \int_0^t \sqrt{1 + X_s^2} dB_s + \int_0^t \big(\sqrt{1 + X_s^2} + \frac{1}{2}X_s\big) ds$$
$$b) \quad X_t = x + \int_0^t \exp(-X_s) dB_s + \frac{1}{2}\int_0^t \exp(-2X_s) ds$$
$$c) \quad X_t = x + \frac{1}{3}\int_0^t (X_s)^{1/3} ds + \int_0^t (X_s)^{2/3} dB_s$$

   Hint: assume that $X_t = \varphi(B_t)$ and use Ito formula to obtain an equation for φ.
   In c) you can assume first that $X_t = \varphi(B_t + a(t))$ and after using Ito formula, choose the function $a(t)$ to simplify the differential equation for φ.

   (b) Rewrite the SDE in Stratonovich form.

   Remark in general is not always possible to find an explicit solutions of a SDE.

7. Let $B^{(1)}$ and $B^{(2)}$ two independent Brownian motions under the measure P and let

$$X_t = x^{(0)}t + x^{(1)}B_t^{(1)} + x^{(2)}B_t^{(2)}$$
$$Y_t = y^{(0)}t + y^{(1)}B_t^{(1)} + y^{(2)}B_t^{(2)}$$

where $x^{(i)}, y^{(i)}$ are deterministic constants, $i = 0, 1, 2$.

Using Girsanov theorem, construct a probability measure Q equivalent to P on finite intervals $[0, t]$ such that both X_t and Y_t are Q-martingales.

Under which conditions on the coefficients $x^{(i)}, y^{(i)}$ such Q is unique ?

8. We consider a family of linear SDE in Ito sense

$$X_t = x + \int_0^t X_s \theta ds + \int_0^t X_s \sigma dB_s^\theta$$

where (B_t^θ) is Brownian motion under the measure P^θ. We think as $\sigma \neq 0$ fixed, while $\theta \in \mathbb{R}$ is a parameter. Note that

$$B_t^\theta = B_t^0 - \frac{\theta}{\sigma} t$$

where B_t^0 is a Brownian motion under P^0 which corresponds to the value $\theta = 0$.

a) Compute and the likelihood ratio process

$$Z_t(\theta) = \frac{dP_t^\theta}{dP_t^0}$$

and find a representation as stochastic integral with respect to the integrator (X_t).

b) Show that $Z_t(\theta)$ is a martingale under P^0.

c) Compute the logarithmic derivative

$$S_t(\theta) := \frac{d}{d\theta} \log Z_t(\theta)$$

and show that $S_t(\theta)$ is a martingale under P^θ.

d) Assuming now that the parameter θ is unknown, compute the maximum likelihood estimator $\hat{\theta}_T$ for a given a realization $(X_t(\omega) : t \in [0, T])$. In other words, find the argument $\hat{\theta}$ which maximizes $\log(Z_t(\theta, \omega))$ for the observed realization.