Stochastic analysis, spring 2013, Solutions Exercises-4, 14.02.2013

1. Let $\tau_1(\omega)$ and $\tau_2(\omega)$ stopping times with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in T)$ taking values in T . Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Use the definition of stopping time to show that $\sigma(\omega) = \min(\tau_1(\omega), \tau_2(\omega))$ is a $\mathbb{F}$-stopping time.

   Solution.

   $$\{\sigma \le t\} = \{\tau_1 \le t\} \cap \{\tau_2 \le t\} \in \mathcal{F}_t \quad \forall t$$

2. Let $(M_t : t \in \mathbb{R}^+)$ a $\mathbb{F}$-martingale, and τ a $\mathbb{F}$-stopping time.

   Show that the stopped process $(M_{t\wedge\tau} : t \in \mathbb{R}^+)$

   $$M_t^\tau(\omega) = M_{t\wedge\tau}(\omega) = M_t(\omega)\mathbf{1}(\tau(\omega) > t) + M_{\tau(\omega)}(\omega)\mathbf{1}(\tau(\omega) \le t)$$

   is a $\mathbb{F}$-martingale.

   We have shown this when $T = \mathbb{N}$ is discrete by using the martingale transform. In continuous time we have not yet defined such martingale transforms. Prove the statement directly by using the definitions.

   Solution. This exercise requires Doob's optional sampling theorem and two regularization results from the continuous time theory which will be presented in coming lectures (sorry for that).

   - If τ is an $\mathbb{F}$-stopping time,

     $$\tau_n(\omega) = \inf\big\{k2^{-n} : \ k \in N, \ k2^{-n} \ge \tau(\omega)\big\} \in D_n$$

     is a $\mathbb{F}$-stopping time.

     This follows easily: since τ is a stopping time

     $$\big\{\omega : \ \tau_n(\omega) > t\big\} = \big\{\omega : \tau(\omega) > t_n - 2^{-n}\big\} \in \mathcal{F}_{t_n - 2^{-n}} \subseteq \mathcal{F}_t$$

     where we have defined

     $$t_n(\omega) = \inf\big\{k2^{-n} : \ k \in \mathbb{N}, \ k2^{-n} \ge t\big\} \in D_n$$

     satisfies $t_n - 2^{-n} < t \le t_n$.

     Note that $\tau_n(\omega) \downarrow \tau(\omega)$ and $t_n \downarrow t$ as $n \uparrow \infty$. A bounded stopping time is approximated from above by a sequence of stopping times taking finitely many values.

     Note also that it is not always possible to approximate a stopping from below by a sequence of stopping times. In continuous time such such stopping times are called *predictable* times.

   - Doob's discrete time optional stopping lemma:

     Lemma 1. *If* $0 \le \sigma(\omega) \le \tau(\omega)$ *are* **bounded** *stopping times in the discrete time filtration* $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$*, and* M_t *is a* $\mathbb{F}$*-martingale,*

     $$E_P(M_\tau|\mathcal{F}_\sigma)(\omega) = M_\sigma(\omega) \tag{1}$$

where $\mathcal{F}_\sigma$ is the stopped σ-algebra.

- In continuous time, a martingale has a right continuous modifications with left limits, we say *cadlag*, Therefore we can always assume that all the martingales we use are cadlag.

Note: assuming that the paths are cadlag, it follows that the paths $t \mapsto M_t(\omega)$ are Borel measurable and therefore the composition $\omega \mapsto M_{\tau(\omega)}(\omega)$ is $\mathcal{F}$-measurable.

In particular $M_t(\omega) = \lim_{n\to\infty} M_{t_n}(\omega)$. For $s = t$ there is nothing to prove, since when τ is an $\mathbb{F}$-stopping time,

$$M_{\tau\wedge t}(\omega) = M_\tau(\omega)\mathbf{1}(\tau(\omega) \le t) + M_t\mathbf{1}(\tau(\omega) > t)$$

is $\mathcal{F}_t$ measurable, since $\tau(\omega)\mathbf{1}(\tau(\omega) \le t)$ is $\mathcal{F}_t$-measurable, because $\{r < \tau(\omega) \le s\} \in \mathcal{F}_t$, $\forall\, 0 \le r < s \le t$.

Consider $0 \le s < t$ and let

$$s_n(\omega) = \inf\big\{k2^{-n} :\ k \in N,\ k2^{-n} \ge s\big\} \in D_n$$

a dyadic sequence with $s_n \downarrow s$, $s_n \le t_n$.

Now note that at each dyadic level n

$$E(M_{\tau_n\wedge t_n}|\mathcal{F}_{s_n})(\omega) = M_{\tau_n\wedge s_n}(\omega) \tag{2}$$

follows directly from the discrete time result which we obtained by using the martingale transform representation of the stopped process.

Now $(\tau_n(\omega) \wedge s_n) \downarrow (\tau(\omega) \wedge s)$, and since $u \mapsto M_u(\omega)$ is right-continuous,

$$M_{\tau_n\wedge s_n}(\omega) \longrightarrow M_{\tau\wedge s}(\omega)$$

For the left hand side of (2), note that by applying Doob optional stopping theorem at the dyadic level n for the bounded stopping times

$(\tau_n \wedge t_n) \le (\tau_0 \wedge t_0) \le (t+1)$ we obtain

$$M_{\tau_n\wedge t_n} = E(M_{\tau_0\wedge t_0}|\mathcal{F}_{\tau_n\wedge t_n})$$

where $M_{\tau_0\wedge t_0} \in L^1(P)$, (by the martingale transform representation), and we work in the non-increasing discrete time filtration $(\mathcal{F}_{\tau_k\wedge t_k}; k \in \mathbb{N})$.

This implies that the collection

$$\big\{M_{\tau_n\wedge t_n} : n \in \mathbb{N}\big\}$$

is uniformly integrable, and by Doob martingale backward convergence theorem $M_{\tau_n\wedge t_n}(\omega) \to M_{\tau\wedge t}(\omega)$, not only P-a.s. but also in $L^1(P)$ norm.

We show that $Y_n := E(M_{\tau_n\wedge t_n}|\mathcal{F}_{s_n}) \longrightarrow Y := E(M_{\tau\wedge t}|\mathcal{F}_s)$ by using directly the definition of conditional expectation.

Note that $(Y_n : n \in N)$ is uniformly integrable, because $(M_{\tau_n \wedge t_n} : n \in \mathbb{N})$ is U.I. For $A \in \mathcal{F}_s \subseteq \mathcal{F}_{s_n}$,

$$E\big(Y_n \mathbf{1}_A\big) = E\big(M_{\tau_n \wedge t_n} \mathbf{1}_A\big) \longrightarrow E\big(M_{\tau \wedge t} \mathbf{1}_A\big) = E\big(Y \mathbf{1}_A\big)$$

Note also that $Y_n = M_{\tau_n \wedge s_n}$ is a martingale with time parameter $n \in \mathbb{N}$, and by the backward martingale convergence theorem $Y_n \to Y_\infty = M_{\tau \wedge s}$ P-almost surely and in $L^1(P)$.

Therefore $E\big(Y_n \mathbf{1}_A\big) \to E(Y_\infty \mathbf{1}_A) = E(Y \mathbf{1}_A)$ for all $A \in \mathcal{F}_s$, which implies $Y_\infty = Y$, since Y_∞ is $\mathcal{F}_s$-measurable (assuming that the filtration $\mathbb{F}$ is right-continuous).

It means that $\forall A \in \mathcal{F}_s$

$$E\big(M_{\tau \wedge t} \mathbf{1}_A\big) = E\big(M_{\tau \wedge s} \mathbf{1}_A\big)$$

Note that here we need to assume that the filtration $\mathbb{F}$ is right-continuous, namely

$$\mathcal{F}_s = \mathcal{F}_{s+} := \bigcap_{n \in N} \mathcal{F}_{s_n} \quad . \tag{3}$$

In this way, assuming right-continuity for the filtration, we are able to extend all the results for stopping times in discrete time to continuous time.

3. Let $(M_t(\omega))_{t \in T}$ a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $M_0(\omega) = 0$. Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Define the family of random times $\tau_x : x \in \mathbb{R}$

   $$\tau_x(\omega) = \begin{cases} \inf\{s : M_s \geq x\} & \text{for } x \geq 0 \\ \inf\{s : M_s \leq x\} & \text{for } x < 0 \end{cases}$$

   Show that τ_x is a stopping time.

   Solution For $x > 0, t \in T$,

   $$\{\tau_x > t\} = \{\sup_{s \leq t} M_s < x\} \in \mathcal{F}_t$$

   because by the regularization lemma in the lecture notes a martingale indexed by $\mathbb{R}^+$ has almost surely left and right limits at all times, which means

   $$\sup_{s \leq t} M_s(\omega) = \sup_{s \in [0,t] \cap \mathbb{Q}} M_s(\omega)$$

   is $\mathcal{F}_t$-measurable.

   For $x < 0$ take the infimum.

4. Let

   $$M_t(\omega) = \sum_{s=1}^{t} X_s(\omega)$$

is a binary random walk where $t \in \mathbb{N}$ and $(X_s : s \in \mathbb{N})$ are i.i.d. random variables with

$$P(X_s = \pm 1) = P(X_s = \pm 1 | \mathcal{F}_{s-1}) = 1/2$$

X_s is $\mathcal{F}_s$ measurable and P-independent from $\mathcal{F}_{s-1}$.

- Show that $(M_t)_{t\in\mathbb{N}}$ and $(M_t^2 - t)_{t\in\mathbb{N}}$ are $\mathbb{F}$-martingales.
  Solution

$$\begin{aligned}
&E(M_t|\mathcal{F}_{t-1}) = E_P(M_{t-1} + X_t|\mathcal{F}_{t-1}) = M_{t-1} + 0,\\
&E(M_t^2|\mathcal{F}_{t-1}) = E_P((M_{t-1} + X_t)^2|\mathcal{F}_{t-1}) =\\
&E_P(M_{t-1}^2 + X_t^2 + 2M_{t-1}X_t|\mathcal{F}_{t-1}) = M_{t-1}^2 + E(X_t^2) + 2M_{t-1}E(X_t) = M_{t-1}^2 + 1
\end{aligned}$$

- Consider the stopping time $\sigma(\omega) = \min(\tau_a, \tau_b)$ where $a < 0 < b \in \mathbb{N}$, and the stopped martingales $(M_{t\wedge\sigma})_{t\in\mathbb{N}}$ and $(M_{t\wedge\sigma}^2 - t \wedge \sigma)_{t\in\mathbb{N}}$.
  Show that Doob's martingale convergence theorem applies and

$$\lim_{t\to\infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

  exists P-almost surely.

  Solution The stopped process $(M_{t\wedge\sigma})$ is a martingale taking values in $[a, b]$, therefore it is uniformly integrable (because it is bounded). In particlar it is bounded in $L^1(P)$ and Doob's martingale convergence theorem applies, P-almost surely (and by UI also in $L^1(P)$)

$$\lim_{t\to\infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

  By the way, this implies already that $P(\sigma < \infty) = 1$, since on the set $\{\sigma = \infty\}$ the random walk would continue fluctuating with $(\limsup M_t - \liminf M_t) \geq 1$. In the next step we show that σ is finite in another way.

- Consider now $(M_{t\wedge\sigma}^2 - t \wedge \sigma)$. Use the martingale property together with the reverse Fatou lemma to show that $E(\sigma) < \infty$ which implies $P(\sigma < \infty) = 1$.

  Solution The stopped martingale $(M_{t\wedge\sigma}^2 - t \wedge \sigma)$ is a submartingale bounded from above, since

$$M_{t\wedge\sigma}^2 - t \wedge \sigma \leq (a^2 \vee b^2)$$

  By Doob's convergence theorem it has a limit P a.s. and since $M_{t\wedge\sigma}(\omega) \to M_\sigma(\omega)$,

$$\lim_{t\to\infty} (M_{t\wedge\sigma}^2 - t \wedge \sigma) = (M_\sigma^2 - \sigma) \in L^1(P)$$

  and $M_\sigma^2(\omega)$ is bounded, necessarily $\sigma \in L^1(P)$. In particular $P(\sigma < \infty) = 1$.
  We can see that $\sigma \in L^1(P)$ also by the reverse Fatou lemma:
  which applies since $(M_{t\wedge\sigma}^2 - t \wedge \sigma) \leq (a^2 \vee b^2)$,

$$E(M_\sigma^2 - \sigma) = E(\limsup_{t\to\infty} M_{t\wedge\sigma}^2 - t\wedge\sigma) \geq \limsup_{t\to\infty} E(M_{t\wedge\sigma}^2 - t\wedge\sigma) = 0$$

which implies

$$E(\sigma) \leq E(M_\sigma^2) \leq a^2 \vee b^2 < \infty$$

- For $a < 0 < b \in \mathbb{N}$, compute $P(\tau_a < \tau_b)$.

Hint: a martingale has constant expectation $E_P(M_t) = E_P(M_0)$. This holds also for the stopped martingale $M_t^\tau = M_{t\wedge\tau}$.

Solution

$$0 = E(M_{t\wedge\sigma}) = E(M_\sigma)$$

where $P(\sigma < \infty) = 1$ and by uniform integrability we can take the limit as $t \to \infty$ inside the expactation. Now

$$0 = E(M_\sigma) = aP(M_\sigma = a) + bP(M_\sigma = b) = aP(\tau_a < \tau_b) + b\big(1 - P(\tau_a < \tau_b)\big)$$
$$\Longrightarrow P(\tau_a < \tau_b) = \frac{b}{b-a}$$

5. Let $M_t(\omega) = B_t(\omega)$, $t \in \mathbb{R}^+$, a Brownian motion which is assumed to be $\mathbb{F}$-adapted, and such that for all $0 < s < t$ the increment $(B_t - B_s)$ is P-independent from the σ-algebra $\mathcal{F}_s$.

   Note this since by assumption the Brownian motion is $\mathbb{F}$-adapted, it follows that $\mathcal{F}_t^B = \sigma(B_s : 0 \leq s \leq t) \subseteq \mathcal{F}_t$, which could be strictly bigger.

   - Show that B_t, $M_t = (B_t^2 - t)$ and $Z_t^a = \exp(aB_t - a^2t/2)$ are $\mathbb{F}$-martingales.

     Solution It follows since $(B_t - B_s) \perp\!\!\!\perp \mathcal{F}_s$ for $0 \leq s \leq t$.

$$\begin{aligned}
&E(B_t^2|\mathcal{F}_s) = B_s^2 + E((B_t - B_s)^2|\mathcal{F}_s) + 2B_sE(B_t - B_s|\mathcal{F}_s)\\
&B_s^2 + E((B_t - B_s)^2) + 2B_sE(B_t - B_s) = B_s^2 + (t-s) + 0\\
&E(Z_t|\mathcal{F}_s) = E\big(\exp(aB_t - a^2t/2)\big|\mathcal{F}_s\big) =\\
&E\big(\exp(a(B_t - B_s)\big|\mathcal{F}_s\big)\exp(-a^2(t-s)/2)\exp(aB_s - a^2s/2)\\
&= E\big(\exp(a(B_t - B_s)\big)\exp(-a^2(t-s)/2)Z_s\\
&= E(\exp(a\sqrt{t-s}B_1))\exp(-a^2(t-s)/2)Z_s = Z_s
\end{aligned}$$

     where B_1 has standard Gaussian distribution.

   - Let $\sigma(\omega) = \min(\tau_a(\omega), \tau_b(\omega))$, for $a < 0 < b \in \mathbb{R}$. We will see in the lectures that the Doob martingale convergence theorem applies also to continuous martingales in continuous time. By following the same line of proof as in the random walk case check that $P(\sigma < \infty) = 1$.
   - Let $a < 0 < b \in \mathbb{R}$. Compute $P(\tau_a < \tau_b)$,

Hints

When M is either a Brownian motion or a random walk with bounded jumps, the stopped process $M_{t\wedge\sigma}(\omega)$ is a uniformly bounded martingale. To compute $P(\tau_a < \tau_b)$, use first the martingale property

$$E(M_{t\wedge\sigma}) = E(M_0) = 0,$$

then for $t \to \infty$ use the bounded convergence theorem.

The stopped process $B_{t\wedge\sigma}(\omega)$ is a martingale, and since it takes values in $[a, b]$ it is bounded and therefore uniformly integrable.

By Doob's martingale convergence theorem

$$\lim_{t\to\infty} B_{t\wedge\sigma} = B_\sigma$$

with both P-almost sure convergence and in $L^1(P)$, and $B_\sigma \in L^1(P)$

Also the martingale $(B^2_{t\wedge\sigma} - t \wedge \sigma) \leq a^2 \vee b^2$ is a submartingale bounded from above, and Doob's convergence theorem applies,

$$(B^2_{t\wedge\sigma} - t \wedge \sigma) \to (B^2_\sigma - \sigma)$$

P-almost surely with $(B^2_\sigma - \sigma) \in L^1(P)$. Since $|B^2_\sigma(\omega)|$ is bounded, it follows that $\sigma \in L^1(P)$ and $P(\sigma < \infty)$.

Now since $\sigma(\omega) < \infty$, with probability one, either $B_\sigma(\omega) = a$, or $B_\sigma(\omega) = b$.

To compute $P(\tau_a < \tau_b)$, use first the martingale property and the bounded convergence theorem

$$0 = E(B_0) = E(B_{t\wedge\sigma}) \to E(B_\sigma) = aP(B_\sigma = a) + b(1 - P(B_\sigma = a))$$

and we obtain the same result as in the random walk case.