Stochastic analysis, 7. exercises

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Exercise 1 Suppose we have an urn which contains at time $t = 0$ two balls, one black and one white. At each time $t \in \mathbb{N}$ we draw uniformly at random from the urn one ball, and put it back together with a new ball of the same colour.

We introduce the random variables

$$X_t(\omega) = 1\{\text{the ball drawn at time } t \text{ is black}\}$$

and denote $S_t = (1 + X_1 + \ldots + X_t)$, $M_t = S_t/(t+2)$, the proportion of black balls in the urn. We use the filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_n = \sigma\{X_s : s \in \mathbb{N}, s \le t\}$.

i) Compute the Doob decomposition of (S_t), $S_t = S_0 + N_t + A_t$, where (N_t) is a martingale and (A_t) is predictable.

ii) Show that (M_t) is a martingale and find the representation of (M_t) as a martingale transform $M_t = (C \cdot N)_t$, where (N_t) is the martingale part of (S_t) and (C_t) is predictable.

iv) Note that the martingale $(M_t)_{t\ge 0}$ is uniformly integrable (Why?). Show that P a.s. and in L^1 exists $M_\infty = \lim_{t\to\infty} M_t$. Compute $E(M_\infty)$.

v) Show that $P(0 < M_\infty < 1) > 0$.

Since $M_\infty(\omega) \in [0,1]$, it is enough to show that $0 < E(M_\infty^2) < E(M_\infty)$ with strict inequalities.

Solution 1 i) We have $A_t = \sum_{i=1}^t (E(S_i|\mathcal{F}_{i-1}) - S_{i-1})$ and $N_t = \sum_{i=1}^t (S_i - E(S_i|\mathcal{F}_{i-1}))$. By the definition of S_t, we can write these as

$$\begin{aligned}
A_t &= \sum_{i=1}^t (E(1 + X_1 + \ldots + X_i|\mathcal{F}_{i-1}) - 1 - X_1 - \ldots - X_{i-1}) \\
&= \sum_{i=1}^t E(X_i|\mathcal{F}_{i-1}) = \sum_{i=1}^t M_{i-1} \\
N_t &= \sum_{i=1}^t (1 + X_1 + \ldots + X_i - E(1 + X_1 + \ldots + X_i|\mathcal{F}_{i-1})) = \\
&= \sum_{i=1}^t (X_i - E(X_i|\mathcal{F}_{i-1})) = \sum_{i=1}^t (X_i - M_{i-1})
\end{aligned}$$

ii) M_t is a martingale since

$$E(M_t|\mathcal{F}_{t-1}) = E(S_t/(t+2)|\mathcal{F}_{t-1}) = \frac{E(S_0 + A_t + N_t|\mathcal{F}_{t-1})}{t+2} = \frac{S_0 + A_t + N_{t-1}}{t+2}$$
$$= \frac{S_{t-1} + M_{t-1}}{t+2} = \frac{\frac{S_{t-1}}{t+1}(t+1) + M_{t-1}}{t+2} = \frac{M_{t-1}(t+1) + M_{t-1}}{t+2} = M_{t-1}.$$

Moreover,

$$M_t = \frac{S_t}{t+2} = \frac{S_0 + A_t + N_t}{t+2} = \frac{S_0 + \sum_{i=1}^{t} M_{i-1} + N_t}{t+2},$$

so

$$(t+2)M_t - \sum_{i=1}^{t} M_{i-1} - S_0 = N_t.$$

Therefore

$$N_t - N_{t-1} = (t+2)M_t - \sum_{i=1}^{t} M_{i-1} - S_0 - (t+1)M_{t-1} + \sum_{i=1}^{t-1} M_{i-1} + S_0 = (t+2)(M_t - M_{t-1}).$$

Summing over t gives us

$$M_n = M_0 + \sum_{t=1}^{n} \frac{N_t - N_{t-1}}{t+2}.$$

iv) The martingale M_t is uniformly integrable, since $M_t < 1$ for all t. For the same reason Doob's martingale convergence theorem applies and $M_t \to M_\infty$ almost surely and in L^1. We have $E(M_\infty) = M_0 = \frac{1}{2}$.

v) To reach a contradiction, assume that $P(0 < M_\infty < 1) = 0$, then by (iv) we must have $P(M_\infty = 0) = P(M_\infty = 1) = \frac{1}{2}$. It follows that $E(M_\infty^2) = \frac{1}{2}$. We will compute $E(M_\infty^2)$ to get a contradiction. We note that M_t^2 is a submartingale. Write the Doob decomposition $M_t^2 = M_0^2 + L_t + P_t$, where L_t is a martingale and P_t is a predictable increasing process,

$$L_t = \sum_{i=1}^{t} (M_i^2 - E(M_i^2|\mathcal{F}_{i-1}))$$

$$P_t = \sum_{i=1}^{t} (E(M_i^2|\mathcal{F}_{i-1}) - M_{i-1}^2)$$
$$= \sum_{i=1}^{t} \left(\frac{E(S_i^2|\mathcal{F}_{i-1})}{(i+2)^2} - \frac{S_{i-1}^2}{(i+1)^2} \right)$$
$$= \sum_{i=1}^{t} \left(\frac{S_{i-1}^2 + 2S_{i-1}E(X_i|\mathcal{F}_{i-1}) + E(X_i^2|\mathcal{F}_{i-1})}{(i+2)^2} - \frac{S_{i-1}^2}{(i+1)^2} \right)$$
$$= \sum_{i=1}^{t} \left(\frac{(i+1)^2 S_{i-1}^2 + 2(i+1)^2 S_{i-1} M_{i-1} + (i+1)^2 M_{i-1} - (i+2)^2 S_{i-1}^2}{(i+1)^2 (i+2)^2} \right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{t} \left(\frac{(-2i-3)S_{i-1}^2 + 2(i+1)S_{i-1}^2 + (i+1)S_{i-1}}{(i+1)^2(i+2)^2} \right) \\
&= \sum_{i=1}^{t} \left(\frac{(i+1)S_{i-1} - S_{i-1}^2}{(i+1)^2(i+2)^2} \right) \\
&= \sum_{i=1}^{t} \left(\frac{S_{i-1}}{(i+1)(i+2)^2} - \frac{S_{i-1}^2}{(i+1)^2(i+2)^2} \right) \\
&= \sum_{i=1}^{t} \left(\frac{M_{i-1} - M_{i-1}^2}{(i+2)^2} \right).
\end{aligned}
$$

Thus we get a recurrence relation for $E(M_t^2)$, namely

$$
E(M_t^2) = E(P_t) + E(M_0^2) = \frac{1}{4} + \sum_{i=1}^{t} \left(\frac{E(M_{i-1}) - E(M_{i-1}^2)}{(i+2)^2} \right) = \frac{1}{4} + \sum_{i=1}^{t} \left(\frac{1}{2(i+2)^2} - \frac{E(M_{i-1}^2)}{(i+2)^2} \right).
$$

Write $a_n = E(M_n^2)$. Then it follows that

$$
a_n - a_{n-1} = \frac{1}{2(n+2)^2} - \frac{1}{(i+2)^2} a_{n-1},
$$

which can also be written as

$$
a_n = \frac{1}{2(n+2)^2} + \frac{(n+1)(n+3)}{(n+2)^2} a_{n-1}, \quad a_0 = 1/4.
$$

We will show by induction that $a_n = \frac{1}{3} - \frac{1}{6(n+2)}$. This clearly holds for a_0. Assuming that it holds for a_n, we get

$$
\begin{aligned}
a_{n+1} &= \frac{1}{2(n+3)^2} + \frac{(n+2)(n+4)}{(n+3)^2} \left(\frac{1}{3} - \frac{1}{6(n+2)} \right) \\
&= \frac{1}{2(n+3)^2} + \frac{(n+2)(n+4)(2(n+2)-1)}{6(n+2)(n+3)^2} \\
&= \frac{3(n+2) + (n+2)(n+4)(2(n+2)-1)}{6(n+2)(n+3)^2} \\
&= \frac{3(n+2) + 2(n+2)^2(n+4) - (n+2)(n+4)}{6(n+2)(n+3)^2} \\
&= \frac{3(n+2) + 2(n+2)^2(n+4) - (n+2)}{6(n+2)(n+3)^2} - \frac{1}{6(n+3)} \\
&= \frac{2(n+2)(1 + (n+2)(n+4))}{6(n+2)(n+3)^2} - \frac{1}{6(n+3)} \\
&= \frac{1}{3} - \frac{1}{6(n+3)}.
\end{aligned}
$$

Therefore it follows that $E(M_\infty^2) = \lim_{n\to\infty} a_n = 1/3 < 1/2$, which is a contradiction.

Exercise 2 Consider an i.i.d. random sequence $(U_t : t \in \mathbb{N})$ with uniform distribution on $[0,1]$, $P(U_1 \in dx) = 1_{[0,1]}(x)dx$. Note that $E(U_t) = 1/2$.

Consider also the random variable $-\log(U_1(\omega))$ which is 1-exponential w.r.t. P.

$$P(-\log(U_1) > x) = \begin{cases} \exp(-x), & \text{if } x \geq 0 \\ 1, & \text{if } x < 0 \end{cases}$$

$-\log(U_1) \in L^1(P)$ with $E(-\log(U_1)) = 1$.

(a) Let $Z_0 = 1$, and

$$Z_t(\omega) = 2^t \prod_{s=1}^{t} U_s(\omega)$$

Show that (Z_t) is a martingale in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, with $\mathcal{F}_t = \sigma(Z_1, ..., Z_t) = \sigma(U_1, ..., U_t)$.

(b) Show that $E(Z_t) = 1$.

(c) Show that the limit $Z_\infty(\omega) = \lim_{t\to\infty} Z_t(\omega)$ exists P-almost surely.

(d) Show that

$$Z_\infty(\omega) = 0 \quad \text{P-a.s.}$$

(e) Show that the martingale Z_t is not uniformly integrable.

(f) Show that $\log(Z_t(\omega))$ is a supermartingale, does it satisfy the assumptions of Doob's martingale convergence theorem?

(g) At every time $t \in \mathbb{N}$, define the probability measure

$$Q_t(A) = E(Z_t 1_A) \quad \forall A \in \mathcal{F}_t$$

on the probability space $(\Omega, \mathcal{F}_t)$.

Show that the random variables $(U_1, ..., U_t)$ are i.i.d. also under Q_t, compute their probability density under Q_t.

Solution 2 (a) Z_t is clearly integrable. Moreover,

$$E(Z_t|\mathcal{F}_{t-1}) = 2^t \prod_{s=1}^{t-1} U_s E(U_t|\mathcal{F}_{t-1}) = 2^{t-1} \prod_{s=1}^{t-1} U_s$$

by independence.

(b) Since Z_t is a martingale, we have $E(Z_t) = Z_0 = 1$.

(c) The martingale Z_t is bounded from below, so by Doob's martingale convergence theorem $Z_\infty(\omega) = \lim_{t\to\infty} Z_t(\omega)$ exists P-almost surely.

(d) Let $\varepsilon > 0$. We will show that

$$\sum_{t=1}^{\infty} P(Z_t > \varepsilon) < \infty.$$

Then by Borel-Cantelli lemma, $Z_t(\omega) \leq \varepsilon$ almost surely for t large enough. Because ε is arbitrary, the result will follow. Now

$$P(Z_t > \varepsilon) = P\left(\prod_{s=1}^{t} U_s > \frac{\varepsilon}{2^t}\right) = P\left(-\sum_{s=1}^{t} \log U_s < \log \frac{2^t}{\varepsilon}\right).$$

Recalling that the sum of i.i.d. exponentially distributed random variables has Gaussian distribution, we see that

$$P(Z_t > \varepsilon) = \int_0^{\log \frac{2^t}{\varepsilon}} \frac{x^{t-1}e^{-x}}{(t-1)!}\,dx.$$

Integration by parts gives us

$$\int_0^{\log \frac{2^t}{\varepsilon}} \frac{x^{t-1}e^{-x}}{(t-1)!}\,dx = -\Big[_0^{\log \frac{2^t}{\varepsilon}} \frac{x^{t-1}e^{-x}}{(t-1)!}\Big] + \int_0^{\log \frac{2^t}{\varepsilon}} \frac{x^{t-2}e^{-x}}{(t-2)!}\,dx,$$

and by iterating this we have

$$\int_0^{\log \frac{2^t}{\varepsilon}} \frac{x^{t-1}e^{-x}}{(t-1)!}\,dx = -\Big[_0^{\log \frac{2^t}{\varepsilon}} e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{t-1}}{(t-1)!}\right)\Big].$$

Therefore

$$P(Z_t > \varepsilon) = 1 - \frac{\varepsilon}{2^t}\left(1 + \log\frac{2^t}{\varepsilon} + \ldots + \frac{\left(\log \frac{2^t}{\varepsilon}\right)^{t-1}}{(t-1)!}\right) = \frac{\varepsilon}{2^t}\sum_{i=t}^{\infty} \frac{\left(\log \frac{2^t}{\varepsilon}\right)^{i}}{i!}.$$

Now, we can change the order of summation to get

$$\sum_{t=1}^{\infty} P(Z_t > \varepsilon) = \sum_{i=1}^{\infty}\sum_{t=1}^{i} \frac{\varepsilon\left(\log \frac{2^t}{\varepsilon}\right)^{i}}{2^t i!}.$$

Moreover, for large i, $\frac{\varepsilon\left(\log \frac{2^t}{\varepsilon}\right)^{i}}{2^t i!}$ increases as t increases from 1 to i:

$$\begin{aligned}
\frac{\frac{\left(\log \frac{2^{t+1}}{\varepsilon}\right)^{i}}{2^{t+1}}}{\frac{\left(\log \frac{2^{t}}{\varepsilon}\right)^{i}}{2^{t}}} &= \frac{1}{2}\left(\frac{(t+1)\log 2 - \log\varepsilon}{t\log 2 - \log\varepsilon}\right)^{i} \\
&\geq \frac{1}{2}\left(1 + \frac{\log 2}{t\log 2 - \log\varepsilon}\right)^{i} \\
&\geq \frac{1}{2}\left(1 + \frac{\log 2}{i\log 2 - \log\varepsilon}\right)^{i} \\
&= \frac{1}{2}\left(1 + \frac{1}{i - \frac{\log\varepsilon}{\log 2}}\right)^{i - \frac{\log\varepsilon}{\log 2}}\left(1 + \frac{1}{i - \frac{\log\varepsilon}{\log 2}}\right)^{\frac{\log\varepsilon}{\log 2}} \to \frac{e}{2} > 1 \quad \text{as } i \to \infty
\end{aligned}$$

It follows that we have the estimate

$$\sum_{t=1}^{\infty} P(Z_t > \varepsilon) \lesssim \sum_{i=1}^{\infty} \frac{\left(\log \frac{2^i}{\varepsilon}\right)^i}{2^i (i-1)!}.$$

Also for large i, $\log \frac{2^i}{\varepsilon} \leq ic$, where $\log 2 < c < 2/e$. (The chosen upper bound will become clear soon.) Thus we only have to show that

$$\sum_{i=1}^{\infty} \frac{(ic/2)^i}{(i-1)!} < \infty.$$

To do this, we will approximate the logarithm of each term:

$$\begin{aligned} i\log(ic/2) - \sum_{j=1}^{i-1} \log j &\leq i \log i + i\log(c/2) - \int_1^{i-1} \log x \, dx \\ &\leq i \log i + i \log(c/2) - (i-1)\log(i-1) + (i-1) + 1 \\ &= i \log \frac{i}{i-1} + i(1 + \log(c/2)) + \log(i-1) \end{aligned}$$

Now $i \log \frac{i}{i-1} \to 1$ as $i \to \infty$ and the linear term $i(1 + \log(c/2))$ dominates $\log(i-1)$ with $1 + \log(c/2) < 0$, so

$$\frac{(ic/2)^i}{(i-1)!} \lesssim Ce^{-Di}$$

for some constants $C, D > 0$ and the series converges.

(e) Since $E(Z_\infty) = 0 \neq \lim_{t\to\infty} E(Z_t) = 1$, the martingale cannot be uniformly integrable.

(f) Because $\log(Z_t) = t\log 2 + \sum_{i=1}^{t} \log U_i$, and because $\int_0^1 \log x \, dx < \infty$, $\log(Z_t)$ is integrable. Moreover, we have

$$\begin{aligned} E(\log(Z_t)|\mathcal{F}_{t-1}) &= t\log 2 + \sum_{i=1}^{t} E(\log U_i|\mathcal{F}_{t-1}) \leq t \log 2 + \sum_{i=1}^{t-1} \log U_i - \log 2 \\ &= (t-1)\log 2 + \sum_{i=1}^{t-1} \log U_i = \log\left(2^{t-1} \prod_{i=1}^{t-1} U_i\right) = \log(Z_{t-1}), \end{aligned}$$

so $\log(Z_t)$ is a supermartingale. It does not satisfy the assumptions of Doob's martingale convergence theorem, since

$$t \log 2 - t = E(\log(Z_t)) = E(\log(Z_t)^+) - E(\log(Z_t)^-),$$

so $E(\log(Z_t)^-) = E(\log(Z_t)^+) + t(1 - \log 2) \geq t(1 - \log 2) \to \infty$.

(g) Suppose that $1 \leq i < j \leq t$. We have

$$\begin{aligned} Q_t(\{U_i < a\} \cap \{U_j < b\}) &= \int Z_t 1_{\{U_i<a\}} 1_{\{U_j<b\}} \, dP = 2^t \int U_1 ... U_t 1_{\{U_i<a\}} 1_{\{U_j<b\}} \, dP \\ &= 4 \int U_i U_j 1_{\{U_i<a\}} 1_{\{U_j<b\}} \, dP = 4 \int U_i 1_{\{U_i<a\}} \, dP \int U_j 1_{\{U_i<b\}} \, dP \\ &= \int Z_t 1_{\{U_i<a\}} \, dP \int Z_t 1_{\{U_j<b\}} \, dP = Q_t(\{U_i < a\}) Q_t(\{U_j < b\}), \end{aligned}$$

so U_i and U_j are independent. Their cumulative distribution function is

$$U(x) = Q_t(\{U_i < a\}) = 2\int U_i 1_{\{U_i<a\}}\, dP = 2\int_0^1 x 1_{\{x<a\}}\, dx = 2\int_0^a x\, dx = a^2,$$

so the probability density is $u(x) = 2x$.

Exercise 3 Consider a function $f \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$. Define the σ-algebra

$$\mathcal{F}_k = \sigma\{Q_{k,z} = (z2^{-k}, (z+1)2^{-k}], z \in \mathbb{Z}^d\} \subset \mathcal{B}(\mathbb{R}^d), \quad k \in \mathbb{Z}$$

and the two sided filtration $\mathbb{F} = (\mathcal{F}_k : k \in \mathbb{Z})$ where the dyadic cubes $(Q_{k,z} : z \in \mathbb{Z}^d)$ form a partition of $\mathbb{R}^d$, and the functions

$$f_k(x) = \sum_{z\in\mathbb{Z}^d} 1(x \in Q_{k,z}) \frac{1}{|Q_{k,z}|} \int_{Q_{k,z}} f(y)\, dy$$

where for $k \in \mathbb{Z}$, $|Q_{k,z}| = 2^{-kd}$ is the Lebesgue measure of the d-dimensional dyadic cube.

Show that $f_k(x)$ is an $\mathbb{F}$-martingale w.r.t. Lebesgue measure.

Show that $\lim_{k\to-\infty} f_k(x) = 0$ both almost surely and in $L^1(\mathbb{R}^d)$-sense.

Define the *maximal function*

$$f^{\square}(x) := \sup_{k\in\mathbb{Z}} f_k(x).$$

Show that for $1 < p < \infty$

$$\|f^{\square}(x)\|_{L^p(\mathbb{R}^d)} \le \frac{p}{p-1} \sup_{k\in\mathbb{Z}} \|f_k\|_{L^p(\mathbb{R}^d)} \le \frac{p}{p-1} \|f\|_{L^p(\mathbb{R}^d)}$$

and

$$cP(|f^{\square}(x)| > c) \le \sup_{k\in\mathbb{Z}} \|f_k\|_{L^1(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)}$$

Solution 3 We will proceed lemma by lemma.

Lemma 1 We have $\|f_k\|_p \le \|f\|_p$ for all $k \in \mathbb{Z}$ and $1 \le p < \infty$.

Proof. We have

$$\begin{aligned}\|f_k\|_p^p &= \sum_{z\in\mathbb{Z}^d} |Q_{k,z}| \cdot \frac{1}{|Q_{k,z}|^p} \left| \int_{Q_{k,z}} f(y)\, dy \right|^p \\ &\le \sum_{z\in\mathbb{Z}^d} \frac{1}{|Q_{k,z}|^{p-1}} \left(\int_{Q_{k,z}} |f(y)|\, dy \right)^p\end{aligned}$$

If $p = 1$, then this is just $\|f\|_1$, proving the claim. Otherwise we apply Hölder's inequality and get

$$\|f_k\|_p^p \leq \sum_{z\in\mathbb{Z}^d} \frac{1}{|Q_{k,z}|^{p-1}} \int_{Q_{k,z}} |f(y)|^p \, dy |Q_{k,z}|^{p/q} = \|f\|_p^p.$$

□

The above lemma shows in particular that each f_k is integrable.

Lemma 2 For each $k \in \mathbb{Z}$, $E(f_k) = E(f)$. Moreover, $E(f|\mathcal{F}_k) = f_k$.

Proof. We have

$$\int_{\mathbb{R}^d} f_k(x)\, dx = \sum_{z\in\mathbb{Z}^d} |Q_{k,z}| \cdot \frac{1}{|Q_{k,z}|} \int_{Q_{k,z}} f(y)\, dy = \int_{\mathbb{R}^d} f(x)\, dx.$$

Clearly f_k is $\mathcal{F}_k$ measurable. In addition if $A \in \mathcal{F}_k$, then A is a union of disjoint dyadic cubes Q_i. We have

$$\int_A f(x)\, dx = \sum_{Q_i} \int_{Q_i} f(x)\, dx = \sum_{Q_i} |Q_i| f_k(z_i) = \sum_{Q_i} \int_{Q_i} f_k(y)\, dy = \int_A f_k(y)\, dy,$$

where z_i is any point in Q_i. Thus by the uniqueness of conditional expectation, $E(f|\mathcal{F}_k) = f_k$. □

Lemma 3 The sequence f_k is a martingale.

Proof. By previous lemma this follows as usual from the tower property of the conditional expectation

$$E(f_{k+1}|\mathcal{F}_k) = E(E(f|\mathcal{F}_{k+1})|\mathcal{F}_k) = f_k.$$ □

Lemma 4 We have $f_k \to 0$ almost everywhere as $k \to -\infty$.

Proof. For every $x \in \mathbb{R}^d$,

$$|f_k(x)| \leq \sum_{z\in\mathbb{Z}^d} 1(x \in Q_{k,z}) \frac{1}{|Q_{k,z}|} \int_{Q_{k,z}} |f(y)|\, dy \leq \frac{1}{2^{-kd}} \|f\|_{L^1}.$$

The right hand side goes to 0 as $k \to -\infty$. □

Lemma 5 We have $f_n \to f$ almost surely and in L^1 as $n \to \infty$.

Proof. Almost surely convergence follows either by using Lebesgue differentiation theorem, or by the martingale convergence theorem. To show convergence in L^1, we split the integral into two parts. Let $\varepsilon > 0$. Then there exists a (large) dyadic cube Q, such that $\int_{\mathbb{R}^d\setminus Q} |f(x)|\, dx < \varepsilon$. Then also

$$\int_{\mathbb{R}^d\setminus Q} |f_k(x) - f(x)|\, dx \leq \int_{\mathbb{R}^d\setminus Q} |f_k(x)| + |f(x)|\, dx < 2\varepsilon,$$

because $\int_{\mathbb{R}^d\setminus Q} |f_k(x)|\, dx \leq \int_{\mathbb{R}^d\setminus Q} |f(x)|\, dx$ when k is large enough so that the cubes on the level k tile the set $\mathbb{R}^d \setminus Q$. (It is enough to assume that k is at least the level on which Q is.) On the other hand, Q has finite measure (which we can normalize to be 1), and $f_k|Q$ is uniformly integrable because $f_k = E(f|\mathcal{F}_k)$. Therefore $f_k|Q \to f|Q$ in L^1. The result follows since $\varepsilon > 0$ was arbitrary, and

$$\int_{\mathbb{R}^d} |f_k(x) - f(x)|\,dx \leq \int_Q |f_k(x) - f(x)|\,dx + \int_{\mathbb{R}^d \setminus Q} |f_k(x) - f(x)|\,dx < 3\varepsilon$$

for k large enough. □

Notice that since f is a martingale, $|f|$ is a submartingale. Define

$$f^{\square}_{n_0,n}(x) = \sup_{n_0 \leq t \leq n} |f_t(x)|.$$

Then for all $n_0 < n \in \mathbb{Z}$, by Theorem 18,

$$cP(f^{\square}_{n_0,n} \geq c) \leq E(|f_n| 1(f^{\square}_{n_0,n} > c)).$$

If we now let $n_0 \to -\infty$, we get

$$cP(f^{\square}_{-\infty,n} \geq c) \leq E(|f_n| 1(f^{\square}_{-\infty,n} > c)). \tag{1}$$

We can use Lemma 18 to get

$$\|f^{\square}_{-\infty,n}\|_p \leq \frac{p}{p-1} \|f_n\|_p,$$

and taking supremum on both sides leaves us with

$$\|f^{\square}\|_p \leq \frac{p}{p-1} \sup_{k \in \mathbb{Z}} \|f_k\|_p \leq \frac{p}{p-1} \|f\|_p.$$

Taking supremums in (1) gives us

$$cP(f^{\square} \geq c) \leq \sup_{k \in \mathbb{Z}} E(|f_n| 1(f^{\square}_{-\infty,n} > c)) \leq \sup_{k \in \mathbb{Z}} E(|f_n|) \leq E(|f|).$$