Stochastic analysis, 6. exercises

Janne Junnila

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Exercise 1 Show that Z_t is a martingale for $\mu = 1$, a supermartingale for $\mu < 1$ and a submartingale for $\mu > 1$.

Solution 1 We show that $E(Z_t) < \infty$ for all t: By induction

$$\begin{aligned} E(Z_t) &= \int_\Omega Z_t(\omega)\,dP = \int_\Omega \sum_{i=1}^{Z_{t-1}(\omega)} Y_{i,t}(\omega)\,dP = \sum_{k=0}^{\infty} \int_{\{Z_{t-1}(\omega)=k\}} \sum_{i=1}^{k} Y_{i,t}(\omega)\,dP \\ &= \sum_{k=0}^{\infty} k\mu P(Z_{t-1} = k) = \mu E(Z_{t-1}) < \infty. \end{aligned}$$

By definition

$$E(Z_{t+1}|\mathcal{F}_t) = E(\sum_{i=1}^{Z_t} Y_{i,t+1}|\mathcal{F}_t) = \sum_{i=1}^{Z_t} E(Y_{i,t+1}|\mathcal{F}_t) = \sum_{i=1}^{Z_t} E(Y_{i,t+1}) = \mu Z_t.$$

Clearly if $\mu = 1$, we have a martingale, if $\mu \le 1$, we have a supermartingale, and if $\mu \ge 1$, we have a submartingale.

Exercise 2 For $\mu \neq 1$, write the Doob decomposition of Z_t and compute $E(Z_t)$.

Solution 2 The Doob decomposition can be defined as $Z_t = M_t + A_t$, where

$$\begin{aligned} M_t &= Z_0 + \sum_{k=1}^{t} (Z_k - E(Z_k|\mathcal{F}_{k-1})) = Z_0 + \sum_{k=1}^{t} (Z_k - \mu Z_{k-1}) \\ A_t &= \sum_{k=1}^{t} (E(Z_k|\mathcal{F}_{k-1}) - Z_{k-1}) = \sum_{k=1}^{t} (\mu - 1) Z_{k-1}. \end{aligned}$$

We already calculated $E(Z_t)$ in the first exercise, but lets do it with the Doob decomposition.

Now $E(M_t) = Z_0 = 1$ for all t, because M is a martingale, and

$$E(A_t) = \sum_{k=1}^{t} (\mu - 1) E(Z_{k-1}) = \sum_{k=1}^{t} (\mu - 1)(1 + E(A_{k-1})).$$

We will prove by induction that $E(A_t) = \mu^t - 1$ for $t \ge 1$. When $t = 1$, this is clearly true, since

$$E(A_1) = E((\mu - 1)(1 + E(A_0))) = E(\mu - 1) = \mu - 1.$$

Suppose that the claim holds for $1, 2, ..., t-1$. Then

$$E(A_t) = (\mu - 1)\sum_{k=1}^{t}(1 + \mu^{k-1} - 1) = (\mu - 1)\frac{\mu^t - 1}{\mu - 1} = \mu^t - 1,$$

so the claim holds for t, too. Therefore $E(Z_t) = E(M_t) + E(A_t) = \mu^t$.

Exercise 3 Assume that $\mu \leq 1$, and that the offspring distribution is non-trivial, meaning that $0 \leq P(Y = 1) < 1$. The case $P(Y = 1) = 1$ is trivial, nothing happens, the size of the population is constant.

Show that when $\mu \leq 1$ (subcritical and critical cases)

$$\lim_{t \to \infty} Z_t(\omega) = 0 \quad P \text{ a.s.}$$

Solution 3 Notice that Z_t is a non-negative supermartingale. Therefore Doob's forward convergence theorem applies and there is a limit Z_∞ almost surely that is integrable. Now

$$P(Z_\infty = 0 | Z_1 = n) = P(Z_\infty = 0)^n, \tag{1}$$

because for extinction we need all the descendants of the n individuals to become extinct, and by independence the probability for that to happen for a single individual is the same as that happening to the first individual. Let $q = P(Z_\infty = 0)$. Then by (1) we have

$$P(Z_\infty = 0 | \sigma(Z_1)) = q^{Z_1},$$

and hence, by taking expectation,

$$q = E(q^{Z_1}) = E(q^Y),$$

where $P(Y = n) = \pi(n)$. Because $\mu \leq 1$ and $P(Y = 1) < 1$, we must have $P(Y = 0) > 0$. Therefore also $q = P(Z_\infty = 0) \geq P(Y = 0) > 0$. Clearly $q = 1$ satisfies the above equation. We will show that there are no other solutions.

Notice that the function $f(q, \omega) = q^{Y(\omega)}$ is integrable for every $q \in (0, 1)$, and it has q-derivatives for almost every ω, so that $\frac{\partial f}{\partial q}(q, \omega) = Y(\omega) q^{Y(\omega) - 1}$. Moreover these derivatives are bounded by the function $Y(\omega)$, which is integrable. Therefore we can differentiate under the sum, and get for $\varphi(q) = E(q^Y) - q$ that

$$\varphi'(q) = -1 + \sum_{y=1}^{\infty} y q^{y-1} \pi(y).$$

Now $\sum_{y=1}^{\infty} y q^{y-1}\pi(y) \leq \sum_{y=1}^{\infty} y \pi(y) = \mu < 1$, so φ is strictly decreasing. Therefore $\varphi(q) = 0$ if and only if $q = 1$.

Exercise 4 In the critical case $\mu = 1$, show that the martingale $(Z_t : t \in \mathbb{N})$ is not uniformly integrable.

Solution 4 Assume that Z_t were U.I. Then $Z_t \to Z_\infty$ in L^1. But this is a contradiction, since by above $Z_\infty = 0$ almost everywhere and on the other hand $E(Z_t) = 1$.

Exercise 5 The next exercises concern the supercritical case $\mu \in (1, \infty)$.

Show that

$$W_t = Z_t(\omega)\mu^{-t}$$

is a martingale.

Solution 5 Notice that by the first exercise

$$E(W_t|\mathcal{F}_{t-1}) = \mu^{-t}E(Z_t|\mathcal{F}_{t-1}) = \mu^{-t}\mu Z_{t-1} = \mu^{-(t-1)}Z_{t-1} = W_{t-1}.$$

Exercise 6 Show that P almost surely $\lim_{t\to\infty} W_t \to W_\infty$ with $W_\infty \in L^1(P)$.

Solution 6 This follows from Doob's martingale convergence theorem since the martingale W_t is non-negative.

Exercise 7 Show that

$$E\left(\left\{\sum_{t=1}^{\infty}\frac{1}{\mu^t}\sum_{i=1}^{Z_{t-1}}\left(Y_{i,t}\mathbf{1}(Y_{i,t} \le \mu^t) - E(Y\mathbf{1}(Y \le \mu^t))\right)\right\}^2\right) < \infty.$$

Solution 7 Because the martingale differences are orthogonal, it is enough to show that

$$\sum_{t=1}^{\infty}\frac{1}{\mu^{2t}}E\left(\sum_{i=1}^{Z_{t-1}}\left(Y_{i,t}\mathbf{1}(Y_{i,t} \le \mu^t) - E(Y\mathbf{1}(Y \le \mu^t))\right)\right)^2 < \infty.$$

Now let $a_t = E(Y\mathbf{1}(Y \le \mu^t))$. Then

$$E\left(\left\{\sum_{i=1}^{Z_{t-1}}\left(Y_{i,t}\mathbf{1}(Y_{i,t} \le \mu^t) - E(Y\mathbf{1}(Y \le \mu^t))\right)\right\}^2\right) =$$

$$E(Z_{t-1})E\left((Y_{i,t}\mathbf{1}(Y_{i,t} \le \mu^t) - a_t)^2\right) -$$

$$E(Z_{t-1})(E(Z_{t-1}) - 1)\sum_{1\le i<j\le Z_{t-1}} E\left((Y_{i,t}\mathbf{1}(Y_{i,t} \le \mu^t) - a_t)(Y_{j,t}\mathbf{1}(Y_{j,t} \le \mu^t) - a_t)\right) =$$

$$\mu^{t-1}\operatorname{Var}(Y\mathbf{1}(Y \le \mu^t)).$$

Thus we just have to show that

$$\sum_{t=1}^{\infty}\frac{\operatorname{Var}(Y\mathbf{1}(Y\le\mu^t))}{\mu^t} = \sum_{t=1}^{\infty}\frac{1}{\mu^t}\left(\sum_{i=0}^{\infty}P(Y = i)i^2 - \left(\sum_{i=0}^{\infty}P(Y = i)i\right)^2\right) < \infty.$$

In particular it is enough to show that

$$\sum_{t=1}^{\infty}\sum_{i=0}^{\infty}P(Y = i)\frac{i^2}{\mu^t} < \infty.$$

This follows by exchanging the order of summation:

$$\sum_{t=1}^{\infty}\sum_{i=0}^{\infty} P(Y=i)\frac{i^2}{\mu^t} \le \sum_{i=0}^{\infty}\sum_{t=\lfloor \log_\mu i\rfloor}^{\infty} P(Y=i)\frac{i^2}{\mu^t} = \sum_{i=0}^{\infty} P(Y=i) i^2 \frac{\mu^{-\lfloor \log_\mu i\rfloor}}{1-\mu^{-1}} \le \frac{\mu}{1-\mu^{-1}}\sum_{i=0}^{\infty} P(Y=i) i < \infty.$$

Exercise 8 Show that also, when $1 < E(Y) < \infty$, without any additional assumptions

$$\sum_{t=1}^{\infty} P(\widetilde{W}_t \neq W_t) < \infty$$

and by Borel Cantelli lemma, with probability one $\widetilde{W}_t \neq W_t$ only for finitely many t.

Solution 8 Notice that

$$\begin{aligned} P(\widetilde{W}_t \neq W_t) &= P\left(\sum_{i=1}^{Z_{t-1}} Y_{i,t}\mathbf{1}(Y_{i,t} \le \mu^t) \neq \sum_{i=1}^{Z_{t-1}} Y_{i,t}\right) \\ &\le P\left(Y_{i,t} > \mu^t \text{ for some } i \in [1, Z_{t-1}]\right) \\ &\le \sum_{z=0}^{\infty} P(Z_{t-1}=z) z P(Y > \mu^t). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{t=1}^{\infty} P(\widetilde{W}_t \neq W_t) &\le \sum_{t=1}^{\infty}\sum_{z=0}^{\infty} P(Z_{t-1}=z) z P(Y>\mu^t) \\ &= \sum_{t=1}^{\infty} P(Y>\mu^t) E(Z_{t-1}) = \frac{1}{\mu}\sum_{t=1}^{\infty} P(Y>\mu^t)\mu^t. \end{aligned}$$

But notice that

$$\begin{aligned} \sum_{t=1}^{\infty} P(Y>\mu^t)\mu^t &= \sum_{t=1}^{\infty}\sum_{i=\lfloor \mu^t\rfloor+1}^{\infty} P(Y=i)\mu^t \\ &\le \sum_{i=\lfloor\mu\rfloor+1} \sum_{t=1}^{\left\lfloor \frac{\log i}{\log \mu}\right\rfloor+1} P(Y=i)\mu^t \\ &= \sum_{i=\lfloor\mu\rfloor+1} P(Y=i)\left(\mu+\mu^2+\ldots+\mu^{\left\lfloor \frac{\log i}{\log \mu}\right\rfloor+1}\right) \\ &= \sum_{i=\lfloor\mu\rfloor+1} P(Y=i)\frac{\mu}{\mu-1}\left(\mu^{\left\lfloor \frac{\log i}{\log \mu}\right\rfloor+1}-1\right) \\ &\le \frac{\mu}{\mu-1}\sum_{i=\lfloor\mu\rfloor+1} P(Y=i)(\mu i-1) < \infty. \end{aligned}$$

Exercise 9 Show that the series

$$\sum_{t=1}^{\infty}\mu^{-t}\sum_{i=1}^{Z_{t-1}}\left\{Y_{i,t}\mathbf{1}(Y_{i,t}>\mu^t) - E(Y\mathbf{1}(Y>\mu^t))\right\}$$

converges in L^1 if and only if $E(Y \log Y) < \infty$.

Solution 9 Assume first that $E(Y \log Y) < \infty$. Then

$$\sum_{t=1}^{\infty} \mu^{-t} \sum_{i=1}^{Z_{t-1}} E(Y\mathbf{1}(Y > \mu^t))$$

converges in L^1, since

$$\sum_{t=1}^{\infty} \mu^{-t} E\left(\sum_{i=1}^{Z_{t-1}} E(Y\mathbf{1}(Y > \mu^t))\right) = \sum_{t=1}^{\infty} \mu^{-t} E(Z_{t-1}) E(Y\mathbf{1}(Y > \mu^t)) = \frac{1}{\mu} \sum_{t=1}^{\infty} E(Y\mathbf{1}(Y > \mu^t)),$$

and

$$\sum_{t=1}^{\infty} E(Y\mathbf{1}(Y > \mu^t)) = \sum_{t=1}^{\infty} \sum_{y=\lfloor \mu^t \rfloor + 1}^{\infty} P(Y = y)y \approx \sum_{y=\lfloor \mu \rfloor} \sum_{t=1}^{\left\lfloor \frac{\log y}{\log \mu} \right\rfloor} P(Y = y)y \approx E(Y \log Y).$$

Now part 8 implies that

$$\sum_{t=1}^{\infty} (W_t - \widetilde{W}_t) = \sum_{t=1}^{\infty} \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-1}} Y_{i,t} \mathbf{1}(Y_{i,t} > \mu^t)$$

converges almost surely. Moreover

$$\sum_{t=1}^{\infty} \frac{1}{\mu^t} E\left(\sum_{i=1}^{Z_{t-1}} Y_{i,t} \mathbf{1}(Y_{i,t} > \mu^t)\right) = \frac{1}{\mu} \sum_{t=1}^{\infty} E(Y\mathbf{1}(Y > \mu^t))$$

converges, so we get the convergence also in L^1.

Suppose then that the martingale W_t converges in L^1. This is equivalent with

$$\sum_{t=1}^{\infty} (W_t - \widetilde{W}_t - R_t) = \sum_{t=1}^{\infty} \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-1}} \left\{ Y_{i,t} \mathbf{1}(Y_{i,t} > \mu^t) - E(Y\mathbf{1}(Y > \mu^t)) \right\}$$

converging in L^1, since the other part

$$\sum_{t=1}^{\infty} (\widetilde{W}_t - W_{t-1} + R_t)$$

converges in L^2 by part 7. In particular there is a subsequence that converges almost surely, and because part 8 tells us that for almost every ω we have $\widetilde{W}_t(\omega) = W_{t-1}(\omega)$ for $t > T(\omega)$, we get that $\sum_{t=1}^{\infty} R_t$ converges almost surely. If we now let $W_*(\omega) = \inf_t W_t(\omega)$, we have $P(W_* > 0) = P(W_\infty > 0) > 0$, since $E(W_\infty) = 1$ by assumption. Thus for almost every ω

$$\infty > \sum_{t=1}^{\infty} R_t(\omega) = \sum_{t=1}^{\infty} \frac{1}{\mu^t} Z_{t-1}(\omega) E(Y\mathbf{1}(Y > \mu^t)) \geq \frac{W_*(\omega)}{\mu} \sum_{t=1}^{\infty} E(Y\mathbf{1}(Y > \mu^t)),$$

so there must exist ω such that $W_*(\omega) > 0$ and the above inequality holds. Then also

$$\sum_{t=1}^{\infty} E(Y\mathbf{1}(Y > \mu^t)) < \infty$$

must hold. We saw above that this is equivalent with $E(Y \log Y) < \infty$.

Exercise 10 Show that when $1 < E(Y) < \infty$, W_t is uniformly integrable if and only if $E(Y \log Y) < \infty$.

Solution 10 We know by parts 7 and 9 that W_t converges in L^1 if and only if $E(Y \log Y) < \infty$. Because convergence in L^1 is equivalent to uniform integrability for martingales satisfying Doob's martingale convergence theorem, the result follows.