Stochastic analysis, spring 2013, Exercises-6, 28.02.2013

A branching process $(Z_t)_{t\in\mathbb{N}}$ with integer values, represents the size of a population evolving randomly in discrete time.

We start with $Z_0(\omega) = 1$ individual at time $t = 0$.

Inductively each of the $Z_{t-1}(\omega)$ individuals in the $(t-1)$ generation has a random number of offspring $Y_{i,t}$. These offspring numbers are independent and identically distributed with law $\pi = (\pi(n) : n = 0, 1, \dots)$,

$\pi(n) = P\big(Y = n\big)$, $Y = Y_{1,1}$.

The size of the new generation at time t is then

$$Z_t(\omega) = \sum_{i=1}^{Z_{t-1}(\omega)} Y_{i,t}(\omega)$$

We assume that the mean offspring number is finite

$$\mu = E_\pi(Y) = \sum_{n=0}^{\infty} n\pi(n) < \infty$$

Note that if $Z_t(\omega) = 0$, then $Z_u(\omega) = 0 \ \forall u \geq t$. In this case we say that the process is extinct. Clearly $P(Z_t = 0) \leq P(Z_u = 0)$ for $t \leq u$.

Note also that $P(Y = 0) > 0$ implies $P(Z_t = 0) > 0$, $\forall t \geq 1$.

Consider the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$ with $\mathcal{F}_t = \sigma(Z_s : 0 \leq s \leq t)$.

Actually we could consider the larger filtration $\mathbb{F}' = (\mathcal{F}'_t : t \in \mathbb{N})$ with

$$\mathcal{F}'_t = \sigma\big(Z_0, Y_{s,i}\mathbf{1}(Z_{s-1} \geq i) : 0 \leq s \leq t,\ i \in \mathbb{N}\big).$$

or $\mathbb{F}'' = (\mathcal{F}''_t : t \in \mathbb{N})$ with

$$\mathcal{F}''_t = \sigma\big(Z_0, Y_{s,i} : 0 \leq s \leq t,\ i \in \mathbb{N}\big).$$

Although $\mathcal{F}_t \subset \mathcal{F}'_t \subset \mathcal{F}''_t$, the martingale properties we use in this exercise for all these filtrations.

1. Show that $Z_t(\omega)$ is a $\mathbb{F}$-martingale, (respectively supermartingale, submartingale) when $\mu = 1$ (respectively $0 \leq \mu < 1$, $1 < \mu < \infty$, in the filtration generated by the process Z itself.

   Solution Note that

$$E(Z_t|\mathcal{F}_{t-1}) = E\left(\sum_{i=1}^{Z_{t-1}} X_{i,t}\middle|\mathcal{F}_{t-1}\right) = \sum_{i=1}^{\infty} E\left(\mathbf{1}(Z_{t-1} \leq i)X_{i,t}\middle|\mathcal{F}_{t-1}\right) =$$
$$= \sum_{i=1}^{\infty} \mathbf{1}(Z_{t-1} \leq i)E\left(X_{i,t}\middle|\mathcal{F}_{t-1}\right) = \sum_{i=1}^{\infty} \mathbf{1}(Z_{t-1} \leq i)E(X_{i,t}) = \mu Z_{t-1}$$

   where we used independence of $X_{i,t}$ from $\mathcal{F}_{t-1}$, and by monotone convergence we can interchange sum and expectation.

2. For $\mu \neq 1$, write the Doob decomposition of the supermartingale (respectively martingale) Z_t as sum of a martingale and a non-increasing (respectively non-decreasing) $\mathbb{F}$-predictable process, and compute the mean $E(Z_t)$ for $t \in \mathbb{N}$.

   Solution

$$Z_t = 1 + \sum_{s=1}^{t} \sum_{i=1}^{Z_{s-1}} (X_{i,s} - 1) =$$

$$1 - (1-\mu) \sum_{s=1}^{t} Z_{s-1} + \sum_{s=1}^{t} \sum_{i=1}^{Z_{s-1}} (X_{i,s} - \mu)$$

   and since the martingale part has zero mean

$$E(Z_t) = 1 + (\mu - 1) \sum_{s=1}^{t} E(Z_{s-1})$$

   this linear difference equation has solution $E(Z_t) = \mu^t$.

3. Assume that $\mu \leq 1$, and that the offspring distribution is non-trivial, meaning that $0 \leq \pi(Y = 1) < 1$. The case $P(Y = 1) = 1$ is trivial, nothing happens, the size of the population is constant.

   Show that when $\mu \leq 1$ (subcritical and critical cases)

$$\lim_{t \to \infty} Z_t(\omega) = 0 \qquad P \text{ a.s.}$$

   Hint: first show that a finite limit $Z_\infty(\omega)$ exists P a.s. with $E(Z_\infty) < \infty$. Use the indepdence of $Y_{1,1}$ from $(Y_{t,i} : t \geq 2, i \in \mathbb{N})$ to prove

$$P(Z_\infty = 0 | Z_1 = n) = P(Z_\infty = 0)^n$$

   where $P(Z_\infty = 0)$ is the probability that the descendance of a single individual becomes extinct.

   . **Solution** Z_t is a non-negative martingale, by Doob's martingale convergence theorem it has P a.s. a finite limit Z_∞.

   By computing first the conditional probability $P(Z_\infty = 0 | \sigma(Z_1))(\omega)$ and taking expectation, show that the unknown $q = P(Z_\infty = 0)$ satisfies the equation

$$q = E_P(q^Y), \quad q \in [\pi(0), 1]$$

   where $P(Y = n) = \pi(n)$ is the offspring distribution.

   Note that since $\mu = E(Y) \leq 1$ and $\pi(1) = P(Y = 1) < 1$, necessarily $\pi(0) = P(Y = 0) > 0$, and $P(Z_\infty = 0) \geq P(Y = 0) > 0$. Therefore $q = 0$ is not a solution.

$q = 1$ is also a solution. We show that there are no other solutions.

Any $0 < q < 1$ is not a solution since the derivative

$$\frac{d}{dq} E_P(q^Y) = E\left(\frac{d}{dq} q^Y \right) = E(Yq^{Y-1}) < E(Y) \le 1$$

with strict inequality since $P(Y = 1) < 1$.

You need to check that it is allowed to take a derivative inside the expectation.

Solution It is enough to check that the derivatives

$$\{Y(\omega)\theta^{Y(\omega)-1} : \theta \in U\}$$

have a common integrable upper bound in an open neighbourhood $U \ni q$. Obviously $Y(\omega)$ is such uniform upper bound for all $q \in (0, 1)$.

This is in contradiction with $E_P(q^Y) = q$ with derivative $\equiv 1$.

4. In the critical case $\mu = 1$, show that the martingale $(Z_t : t \in \mathbb{N})$ is not uniformly integrable

   Solution: Since $0 = E(Z_\infty) < E(Z_t) = E(Z_0) = 1$, $Z_t(\omega) \to Z_\infty(\omega)$ P almost surely but not in $L^1(P)$, therefore uniform integrability does not hold.

   Next we work with the supercritical case, with $\mu = E_P(Y) \in (1, \infty)$.

5. Show that

   $$W_t = Z_t(\omega)\mu^{-t}$$

   is a martingale.

   Solution It follows from $E_P(Z_t|\mathcal{F}_{t-1})(\omega) = Z_{t-1}\mu$.

6. Show that P almost surely $\lim_{t\to\infty} W_t \to W_\infty$ with $W_\infty \in L^1(P)$.

   Solution $W_t(\omega)$ is a non-negative martingale, in particular it is a supermartingale bounded from below and Doob's martingale convergence theorem applies.

7. The next result is a theorem from Kesten and Stigum (1966) which states that W_t is an uniformly integrable martingale if and only if the offspring distribution satisfies

   $$E_P(Y \log(Y)) = 0$$

   where it is understood that $0 \log(0) = \lim_{x\downarrow 0} x \log(x) = 0$.

   Write the increments:

   $$W_t - W_{t-1} = \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-i}} (Y_{t,i} - \mu)$$

and truncate them in the following way: for

$$\widetilde{W}_t = \frac{1}{\mu^t}\sum_{i=1}^{Z_{t-i}} Y_{t,i}\mathbf{1}(Y_{t,i} \le \mu^t),$$

$$R_t = \frac{1}{\mu^t}\sum_{i=1}^{Z_{t-i}} E\Big(Y\mathbf{1}(Y > \mu^t)\Big)$$

We decompose

$$W_t - W_{t-1} = \underbrace{\Big(W_t - \widetilde{W}_t - R_t\Big)}_{\text{I}} + \underbrace{\Big(\widetilde{W}_t + R_t - W_{t-1}\Big)}_{\text{II}} =$$

$$\frac{1}{\mu^t}\sum_{i=1}^{Z_{t-1}}\Big\{Y_{t,i}\mathbf{1}(Y_{t,i} > \mu^t) - E\big(Y\mathbf{1}(Y > \mu_t)\big)\Big\} + \frac{1}{\mu^t}\sum_{i=1}^{Z_{t-1}}\Big\{Y_{t,i}\mathbf{1}(Y_{t,i} \le \mu^t) - E\big(Y\mathbf{1}(Y \le \mu_t)\big)\Big\}$$

where (I) and (II) are martingale differences.
Note that

$$E\bigg(\bigg\{\sum_{t=1}^{\infty}\frac{1}{\mu^t}\sum_{i=1}^{Z_{t-1}}\Big(Y_{t,i}\mathbf{1}(Y_{t,i} \le \mu^t) - E\big(Y\mathbf{1}(Y > \mu^t)\big)\Big)\bigg\}^2\bigg) = \sum_{t=1}^{\infty}\frac{E(Z_{t-1})}{\mu^{2t}}\int_0^{\mu^t} x^2 P(Y \in dx)$$

$$= \frac{1}{\mu^2}\int_0^{\infty}\Big(\sum_{t=1}^{\infty}\mu^{-t}\mathbf{1}(\mu^t > x)\Big)x^2 P(Y \in dx) \le \frac{\mu}{\log\mu}\int_0^{\infty} xP(Y \in dx) = \frac{\mu^2}{\log\mu} < \infty$$

where

$$\sum_{t > \log x/\log\mu}^{\infty}\mu^{-t} \le \int_{\lfloor\frac{\log x}{\log\mu}\rfloor}\exp(-s\log\mu)ds \le \frac{\mu}{x\log\mu}$$

Therefore by summing the increments (I), we obtain a martingale bounded in $L^2(P)$ which is also uniformly integrable.
We show also that, when $1 < E(Y) < \infty$, without any additional assumptions

$$\sum_{t=1}^{\infty} P(\widetilde{W}_t \ne W_t) < \infty$$

In fact

$$\sum_{t=1}^{\infty} P(\widetilde{W}_t \ne W_t) = \sum_{t=1}^{\infty} E_P\Big(P(\widetilde{W}_t \ne W_t|Z_{t-1})\Big)$$

$$\le \sum_{t=1}^{\infty} E_P(Z_{t-1})P(Y > \mu^t) = \frac{1}{\mu}\sum_{t=1}^{\infty}\mu^t P(Y > \mu^t)$$

$$= \frac{1}{\mu}\int_0^{\infty}\Big(\sum_{t=1}^{\infty}\mu^t\mathbf{1}(x > \mu^t)\Big)P(Y \in dx) \le \frac{1}{\mu}\int_0^{\infty}\frac{\mu x - 1}{\mu - 1}P(X \in dx) = 1 + \mu^{-1} < \infty$$

Therefore by Borel Cantelli lemma with probability one $\widetilde{W}_t \ne W_t$ for at most finitely many t.

We show that the series

$$\sum_{t=1}^{\infty} \mu^{-t} \sum_{i=1}^{Z_{t-1}} \Big\{ Y_{t,i} \mathbf{1}(Y_{t,i} > \mu^t) - E_P\big(Y \mathbf{1}(Y > \mu^t)\big) \Big\}$$

converges in $L^1(P)$ if and only if $E_P(Y \log Y) < \infty$.

In fact

$$\sum_{t=1}^{\infty} \mu^{-t} E_P\Bigg(\sum_{i=1}^{Z_{t-1}} Y_{t,i} \mathbf{1}(Y_{t,i} > \mu^t) \Bigg) = \sum_{t=1}^{\infty} \mu^{-t} E_P(Z_{t-1}) E_P\big(Y \mathbf{1}(Y > \mu^t)\big)$$

$$= \frac{1}{\mu} \int_0^{\infty} \sum_{t=1}^{\infty} \mathbf{1}(x > \mu^t) x P(Y \in dx) = \frac{1}{\mu} \int_0^{\infty} \sum_{t=1}^{\infty} \mathbf{1}(t < \frac{\log x}{\log \mu}) x P(Y \in dx)$$

$$\leq \frac{1}{\mu \log \mu} E(Y \log Y) < \infty$$

Next we show that when $(W_t : t \in \mathbb{N})$ is uniformly integrable, then $E_P(Y \log Y) < \infty$.

Since the series of martingale differences is bounded in $L^2(P)$,

$$\sum_{t=1}^{\infty} (\widetilde{W}_t - W_{t-1} + R_t)$$

it is converging P-almost surely and in $L^1(p)$, and $\widetilde{W}_t \neq W_t$ for at most finitely many t,

It follows that the series

$$\sum_{t=1}^{\infty} (W_t - W_{t-1} + R_t)$$

is converging P-almost surely, and since the series

$$\sum_{t=1}^{\infty} (W_t - W_{t-1})$$

is converging P-almost surely by Doob's martingale convergence theorem, we have also almost sure convergence for the series

$$\sum_{t=1}^{\infty} R_t = \frac{1}{\mu} \sum_{t=1}^{\infty} W_{t-1} E(Y \mathbf{1}(Y > \mu^t))$$

Let $\underline{W}(\omega) = \inf_t W_t(\omega)$. Note that $\underline{W}(\omega) = 0 \iff W_\infty = 0$.

By the uniform integrability assumtion $W_t \to W_\infty$ converges also in $L^1(P)$ with $E(W_\infty) = 1$, and necessarily $P(\underline{W} > 0) = P(W_\infty > 0) > 0$,

We have that P a.s.

$$\infty > \sum_{t=1}^{\infty} R_t(\omega) \geq \frac{\underline{W}(\omega)}{\mu} \sum_{t=1}^{\infty} E_P\big(Y \mathbf{1}(Y > \mu^t)\big)$$

Therefore $\exists\omega$ such that $\underline{W}(\omega) > 0$ and

$$\infty > \frac{\mu}{\underline{W}(\omega)} \sum_{t=1}^{\infty} R_t(\omega) \geq \int_0^{\infty} \left(\sum_{t=1}^{\infty} \mathbf{1}(\mu^t < x) \right) x P(Y \in dx) \geq \frac{E_P(Y \log Y)}{\log(\mu)} - \mu$$

Last week we had this problem:

A generalization of a game by Jacob Bernoulli. In this game a fair die is rolled, and if the result is Z_1, then Z_1 dice are rolled. If the total of the Z_1 dice is Z_2, then Z_2 dice are rolled. If the total of the Z_2 dice is Z_3, then Z_3 dice are rolled, and so on. Let $Z_0 \equiv 1$.

In this case $Y_{t,i}(\omega)$ are uniformly distributed in the set $\{1, 2, \dots, 6\}$, with $\mu = E_P(Y) = 7/2 > 1$. Since $P(Y = 0) = 0$, the branching processes never dies.

The condition $E_P(Y \log Y)$ is fullfilled since Y is finite. So $W_t = Z_t \mu^{-t}$ is uniformly integrable and it is convergent both in $L^1(P)$ and P-almost surely to a random variable W_∞ with $E_P(W_\infty) = 1$.