Stochastic analysis, 14. exercises

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Exercise 1 Let $(B_t : t \geq 0)$ be a Brownian motion which generates the filtration $\mathbb{F} = (\mathcal{F}_t^B)$. Compute the Ito representation of the random variables

(a)

$$\int_0^T B_t \, dt$$

(b)

$$\exp\left(\int_0^T h(t) \, dB_t\right)$$

where $h(t) \in L^2([0,T], dt)$ is deterministic.

(c) $\sin(B_t)$ and $\cos(B_t)$.

Solution 1 (a) Integration by parts gives

$$\int_0^T B_t \, dt = TB_T - \int_0^T t \, dB_t = \int_0^T (T-t) \, dB_t,$$

which is the Ito representation since $E\left(\int_0^T B_t \, dt\right) = 0$.

(b) We note that $\exp\left(\int_0^T h(t) \, dB_t - \frac{1}{2}\int_0^T h(t)^2 \, dt\right)$ is a square integrable martingale. Indeed Ito formula gives

$$\exp\left(\int_0^T h(t) \, dB_t - \frac{1}{2}\int_0^T h(t)^2 \, dt\right) = 1 + \int_0^T \exp\left(\int_0^t h(u) \, dB_u - \frac{1}{2}\int_0^t h(u)^2 \, du\right) h(t) \, dB_t$$

and since $\int_0^t h(u) \, dB_u$ is a Gaussian random variable with mean 0 and variance $\int_0^t h(u)^2 \, du$, we have

$$E\left(\int_0^T \exp\left(2\int_0^t h(u) \, dB_u - \int_0^t h(u)^2 \, du\right) h(t)^2 \, dt\right) < \infty.$$

Thus we get

$$\exp\left(\int_0^T h(t)\,dB_t\right) = \exp\left(\frac{1}{2}\int_0^T h(t)^2\,dt\right) + \int_0^T \exp\left(\int_0^t h(u)\,dB_u + \frac{1}{2}\int_t^T h(u)^2\,du\right) h(t)\,dB_t$$

as the Ito representation.

(c) We have

$$\cos(B_T)e^{T/2} + i\sin(B_T)e^{T/2} = e^{iB_T+T/2} = 1 + i\int_0^T e^{iB_t+t/2}\,dB_t = 1 + i\int_0^T \cos(B_t)e^{t/2}\,dB_t - \int_0^T \sin(B_t)e^{t/2}\,dB_t.$$

It follows that we have the Ito representations

$$\cos(B_T) = e^{-T/2} - \int_0^T \sin(B_t)e^{-\frac{T-t}{2}}\,dB_t$$
$$\sin(B_T) = \int_0^T \cos(B_t)e^{-\frac{T-t}{2}}\,dB_t.$$

Exercise 2 Prove the following version of Gronwall's lemma: Let $a_t, b_t\colon \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing functions with $a_0 = 0$. If

$$0 \le x_t \le b_t + \int_0^t x_t da_t \qquad \forall t \ge 0$$

then

$$x_t \le b_t \exp(a_t).$$

Solution 2 Define the function y_t by

$$y_t = \int_0^t e^{a_t - a_s}\,db_s + b_0 e^{a_t}.$$

Then y_t is strict with respect to the given condition:

$$\begin{aligned} b_t + \int_0^t y_s\,da_s &= b_t + \int_0^t \int_0^s e^{a_s - a_u}\,db_u + b_0 e^{a_s}\,da_s \\ &= b_t + \int_0^t \int_u^t e^{a_s - a_u}\,da_s\,db_u + b_0 e^{a_t} - b_0 \\ &= b_t - b_0 + \int_0^t e^{a_t - a_u} - 1\,db_u + b_0 e^{a_t} = y_t. \end{aligned}$$

Moreover y_t satisfies the inequality we want to prove:

$$\int_0^t e^{a_t - a_s}\,db_s + b_0 e^{a_t} \le e^{a_t}\left(\int_0^t 1\,db_s + b_0\right) = e^{a_t} b_t.$$

Finally $x_t \le y_t$ for all t since if we define the function $z_t = y_t - x_t$, then $z_0 \ge 0$ and

$$z_t \geq \int_0^t z_s \, da_s \geq 0.$$

Exercise 3 Write the following Ito stochastic integral as a Stratonovich integral plus a process of finite variation

$$\int_0^t \cos(B_s + W_s) \, dB_s.$$

Write the semimartingale decomposition of the following Stratonovich integrals

$$\int_0^t \exp(B_s + W_s) \circ dW_s$$

where B_t and W_t are independent Brownian motions.

Solution 3 We have

$$\int_0^t \cos(B_s+W_s) \, dB_s = \int_0^t \cos(B_s+W_s) \circ dB_s - \frac{1}{2}[\cos(B_s+W_s), B_s] = \int_0^t \cos(B_s+W_s) \circ dB_s + \frac{1}{2} \int_0^t \sin(B_s+W_s) \, ds,$$

since by Ito formula

$$\cos(B_s + W_s) = 1 - \int_0^t \sin(B_s + W_s) \, dB_s - \int_0^t \sin(B_s + W_s) \, dW_s + \int_0^t \cos(B_s + W_s) \, ds.$$

For the other one,

$$\int_0^t \exp(B_s+W_s) \circ dW_s = \int_0^t \exp(B_s+W_s) \, dW_s + \frac{1}{2}[\exp(B_s+W_s), W_s] = \int_0^t \exp(B_s+W_s) \, dW_s + \frac{1}{2} \int_0^t e^{B_s+W_s} \, ds.$$

Exercise 4 Solve the linear Ito stochastic differential equation

$$X_t^x = x + B_t + \int_0^t \frac{y - X_s^x}{T - s} \, ds, \quad t \in [0, T]$$

where B_t is a Brownian motion.

Write and solve the linear Ito stochastic differential equation for the derivative with respect to the initial value x:

$$\dot{X}_t = \frac{\partial}{\partial x} X_t^x.$$

Solution 4 We have

$$dX_t^x = \frac{y - X_t^x}{T - t} dt + dB_t = \frac{y}{T - t} dt - \frac{X_t^x}{T - t} dt + dB_t.$$

Let us look for a solution of the form $X_t^x = (X_1)_t (X_2)_t$, where

$$d(X_1)_t = -\frac{(X_1)_t}{T-t}dt$$
$$d(X_2)_t = C_t dt + D_t dB_t.$$

Now

$$d(X_1X_2)_t = (X_1)_t d(X_2)_t + (X_2)_t d(X_1)_t + [X_1, X_2]_t$$
$$= (X_1)_t C_t dt + (X_1)_t D_t dB_t - \frac{X_t}{T-t}dt,$$

so we just have to set $C_t = \frac{y}{(T-t)(X_1)_t}$ and $D_t = \frac{1}{(X_1)_t}$. Notice that

$$(X_1)_t = \frac{T-t}{T},$$

so

$$C_t = \frac{yT}{(T-t)^2}, \quad D_t = \frac{T}{T-t}.$$

Hence

$$(X_2)_t = x + \int_0^t \frac{yT}{(T-s)^2}\,ds + \int_0^t \frac{T}{T-s}\,dB_s$$
$$= x + yT\left(\frac{1}{T-t} - \frac{1}{T}\right) + \int_0^t \frac{T}{T-s}\,dB_s$$
$$= x + \frac{yt}{T-t} + \int_0^t \frac{T}{T-s}\,dB_s.$$

Thus

$$X_t = \frac{T-t}{T}x + \frac{yt}{T} + (T-t)\int_0^t \frac{1}{T-s}\,dB_s$$
$$= x + \frac{y-x}{T}t + (T-t)\int_0^t \frac{1}{T-s}\,dB_s.$$

From here we also see that

$$\dot{X}_t = \frac{\partial X_t^x}{\partial x} = \frac{T-t}{T}.$$

Exercise 5 Let $X_t(\omega) \in \mathbb{R}$ be a solution of the stochastic differential equation

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dB_s.$$

Assume that X_s has density $p(y; s, x)$ at every $s > 0$. Find the partial differential equation satisfied by the density.

Write also the partial differential equation for the density of the derivative

$$\dot{X}_t = \frac{\partial}{\partial x} X_t^x$$

assuming that the density exists.

Solution 5 We have $dX_t^x = b(t, X_t^x)dt + \sigma(t, X_t^x)dB_t$. Hence if f is a smooth test function, Ito formula gives us

$$f(X_t^x) = f(x) + \int_0^t f'(X_s^x) b(s, X_s^x) ds + \int_0^t f'(X_s^x)\sigma(s, X_s^x) dB_s + \frac{1}{2}\int_0^t f''(X_s^x)\sigma^2(s, X_s^x) ds.$$

By taking the expectation we get

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y) p(y; t, x) dy &= f(x) + \int_0^t \int_{-\infty}^{\infty} \left(b(s,y) f'(y) + \frac{1}{2}\sigma^2(s,y) f''(y)\right) p(y; s, x)\, dy\, ds \\
&= f(x) + \int_0^t \int_{-\infty}^{\infty} \Bigg(- p(y; s, x) \frac{\partial}{\partial y} b(s,y) - b(s,y) \frac{\partial}{\partial y} p(y; s, x) \\
&\quad + \left(\frac{\partial}{\partial y}\sigma(s,y)\right)^2 p(y; s, x) + \sigma(s,y) \frac{\partial^2}{\partial y^2} \sigma(s,y) p(y; s, x) \\
&\quad + 2\sigma(s,y) \frac{\partial}{\partial y}\sigma(s,y) \frac{\partial}{\partial y} p(y; s, x) + \frac{1}{2}\sigma(s,y)^2 \frac{\partial^2}{\partial y^2} p(y; s, x) \Bigg) f(y)\, dy\, ds
\end{aligned}$$

Thus we get

$$\begin{aligned}
p(y; t, x) = \delta_x + \int_0^t &- \frac{\partial}{\partial y} b(s,y) p(y; s, x) - b(s,y) \frac{\partial}{\partial y} p(y; s, x) \\
&+ \left(\frac{\partial}{\partial y}\sigma(s,y)\right)^2 p(y; s, x) + \sigma(s,y) \frac{\partial^2}{\partial y^2} \sigma(s,y) p(y; s, x) \\
&+ 2\sigma(s,y) \frac{\partial}{\partial y}\sigma(s,y) \frac{\partial}{\partial y} p(y; s, x) + \frac{1}{2}\sigma(s,y)^2 \frac{\partial^2}{\partial y^2} p(y; s, x)\, ds
\end{aligned}$$

Differentiating w.r.t. t gives us the PDE

$$\begin{aligned}
\frac{\partial}{\partial t} p(y; t, x) = &\left(-\frac{\partial}{\partial y} b(t,y) + \left(\frac{\partial}{\partial y}\sigma(t,y)\right)^2 + \sigma(t,y) \frac{\partial^2}{\partial y^2}\sigma(t,y) \right) p(y; t, x) \\
&+ \left(-b(t,y) + 2\sigma(t,y) \frac{\partial}{\partial y}\sigma(t,y) \right) \frac{\partial}{\partial y} p(y; t, x) \\
&+ \frac{1}{2}\sigma(t,y)^2 \frac{\partial^2}{\partial y^2} p(y; t, x)
\end{aligned}$$