Stochastic analysis, 4. exercises

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Exercise 1 Let $\tau_1(\omega)$ and $\tau_2(\omega)$ be stopping times with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in T)$ taking values in T. Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

Use the definition of stopping time to show that $\sigma(\omega) = \min(\tau_1(\omega), \tau_2(\omega))$ is a $\mathbb{F}$-stopping time.

Solution 1 We have

$$\{\sigma(\omega) \le t\} = \{\omega : \tau_1(\omega) \le t \text{ or } \tau_2(\omega) \le t\} = \{\tau_1(\omega) \le t\} \cup \{\tau_2(\omega) \le t\} \in \mathcal{F}_t,$$

so σ is a stopping time.

Exercise 2 Let $(M_t : t \in \mathbb{R}^+)$ be a $\mathbb{F}$-martingale, and τ a $\mathbb{F}$-stopping time.

Show that the stopped process $(M_{t\wedge\tau} : t \in \mathbb{R}^+)$

$$M_t^\tau(\omega) = M_{t\wedge\tau}(\omega) = M_t(\omega)\mathbf{1}(t \le \tau(\omega)) + M_{\tau(\omega)}(\omega)\mathbf{1}(t > \tau(\omega))$$

is a $\mathbb{F}$-martingale.

Solution 2

Exercise 3 Let $(M_t(\omega))_{t\in T}$ be a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $M_0(\omega) = 0$. Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

Define a family of random times $\tau_x : x \in \mathbb{R}$

$$\tau_x(\omega) = \begin{cases} \inf\{s : M_s \ge x\} & \text{for } x \ge 0 \\ \inf\{s : M_s \le x\} & \text{for } x < 0 \end{cases}$$

Show that τ_x is a stopping time.

Solution 3 Assume that $T = \mathbb{N}$. Then if $x \ge 0$

$$\{\tau_x \le t\} = \{\inf\{s : M_s \ge x\} \le t\} = \{M_s \ge x \text{ for some } s \le t\} = \bigcup_{s=1}^{t} \{M_s \ge x\},$$

where $\{M_s \ge x\} \in \mathcal{F}_s \subset \mathcal{F}_t$, so $\{\tau_x \le t\} \in \mathcal{F}_t$. Thus τ_x is a stopping time. Similarly if $x < 0$, then

$$\{\tau_x \le t\} = \{\inf\{s : M_s \le x\} \le t\} = \{M_s \le x \text{ for some } s \le t\} = \bigcup_{s=1}^{t} \{M_s \le x\} \in \mathcal{F}_t.$$

Exercise 4 Let

$$M_t(\omega) = \sum_{s=1}^{t} X_s(\omega)$$

be a binary random walk where $t \in \mathbb{N}$ and $(X_s : s \in \mathbb{N})$ are i.i.d. random variables with

$$P(X_s = \pm 1) = P(X_s = \pm 1 | \mathcal{F}_{s-1}) = 1/2.$$

X_s is $\mathcal{F}_s$ measurable and P-independent from $\mathcal{F}_{s-1}$.

- Show that $(M_t)_{t\in\mathbb{N}}$ and $(M_t^2 - t)_{t\in\mathbb{N}}$ are $\mathbb{F}$-martingales.

- Consider the stopping time $\sigma(\omega) = \min(\tau_a, \tau_b)$ where $a < 0 < b \in \mathbb{N}$, and the stopped martingales $(M_{t\wedge\sigma})_{t\in\mathbb{N}}$ and $(M^2_{t\wedge\sigma} - t \wedge \sigma)_{t\in\mathbb{N}}$.

  Show that Doob's martingale convergence theorem applies and

  $$\lim_{t\to\infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

  exists P-almost surely.

- Consider now $(M^2_{t\wedge\sigma} - t \wedge \sigma)$. Use the martingale property together with the reverse Fatou lemma to show that $E(\sigma) < \infty$ which implies $P(\sigma < \infty) = 1$.

- For $a < 0 < b \in \mathbb{N}$, compute $P(\tau_a < \tau_b)$.

Solution 4 (*M_t and $M_t^2 - t$ are martingales:*) Clearly M_t and $M_t^2 - t$ are bounded by t and $t^2 - t$, so they are integrable. They are also martingales since if $s < t$, then

$$E(M_t|\mathcal{F}_s) = E(M_{t-1} + X_t|\mathcal{F}_s) = M_s + E(X_t|\mathcal{F}_s) = M_s$$

by induction and independence. Similarly

$$\begin{aligned} E(M_t^2 - t|\mathcal{F}_s) &= E(M^2_{t-1} + 2M_{t-1}X_t + X_t^2 - t|\mathcal{F}_s) = M_s^2 - s + E(2M_{t-1}X_t + X_t^2 - 1|F_s) \\ &= M_s^2 - s + 2E(M_{t-1}|F_s)E(X_t|F_s) = M_s^2 - s. \end{aligned}$$

(*Limit of the stopped martingale:*) We have to check that

$$\sup_{t\geq 0} E(M^-_{t\wedge\sigma}) < \infty.$$

But this is clear since $M^-_{t\wedge\sigma} \leq -a$. Hence the limit

$$\lim_{t\to\infty} M_{t\wedge\sigma}$$

exists almost surely. Similarly we see that the limit

$$\lim_{t\to\infty} M^2_{t\wedge\sigma} - t \wedge \sigma$$

exists almost surely because $(M^2_{t\wedge\sigma} - t \wedge \sigma)^+ \leq a^2 + b^2$.

(*We have $E(\sigma) < \infty$:*) We know that $M^2_{t\wedge\sigma} - (t \wedge \sigma)$ is a martingale that almost surely converges to $M^2_\sigma - \sigma$. Thus by reverse Fatou lemma

$$E(M_\sigma^2 - \sigma) \geq \limsup_{t\to\infty} E(M_{t\wedge\sigma}^2 - (t \wedge \sigma)) = 0.$$

In particular then

$$E(\sigma) \leq E(M_\sigma^2) < \infty.$$

(*The probability* $P(\tau_a < \tau_b)$*:*) By Doob's optional stopping theorem we have

$$E(M_\sigma) = E(M_0) = 0 = P(\tau_a < \tau_b)a + (1 - P(\tau_a < \tau_b))b,$$

so $P(\tau_a < \tau_b) = \frac{b}{b-a}$.

Exercise 5 Let $M_t(\omega) = B_t(\omega)$, $t \in \mathbb{R}^+$, a Brownian motion which is assumed to be $\mathbb{F}$-adapted, and such that for all $0 < s < t$ the increment $(B_t - B_s)$ is P-independent from the σ-algebra $\mathcal{F}_s$.

Note that since by assumption the Brownian motion is $\mathbb{F}$-adapted, it follows that $\mathcal{F}_t^B = \sigma(B_s : 0 \leq s \leq t) \subset \mathcal{F}_t$, which could be strictly bigger.

- Show that B_t, $M_t = B_t^2 - t$ and $Z_t^a = \exp(aB_t - \frac{1}{2}a^2 t)$ are $\mathbb{F}$-martingales.
- Let $\sigma(\omega) = \min(\tau_a(\omega), \tau_b(\omega))$, for $a < 0 < b \in \mathbb{R}$. We will see in the lectures that the Doob martingale convergence theorem applies also to continuous martingales in continuous time. By following the same line of proof as in the random walk case check that $P(\sigma < \infty) = 1$.
- Let $a < 0 < b \in \mathbb{R}$. Compute $P(\tau_a < \tau_b)$.

Solution 5 (B_t*,* $B_t^2 - t$ *and* $Z_t^a = \exp(aB_t - \frac{1}{2}a^2 t)$ *are martingales:*) Because B_t has Gaussian distribution, it is integrable. Moreover

$$E(B_t|\mathcal{F}_s) = E(B_t - B_s + B_s|\mathcal{F}_s) = E(B_t - B_s) + B_s = B_s.$$

We also have

$$E(B_t^2 - t|\mathcal{F}_s) = E((B_t - B_s)^2 + 2B_tB_s - B_s^2 - t|\mathcal{F}_s) = E((B_t - B_s)^2) + 2B_s^2 - B_s^2 - t = B_s^2 - s$$

and

$$E(e^{aB_t - \frac{1}{2}a^2 t}|\mathcal{F}_s) = E(e^{a(B_t - B_s)}e^{aB_s - \frac{1}{2}a^2 t}|\mathcal{F}_s) = E(e^{a(B_t - B_s)})e^{aB_s - \frac{1}{2}a^2 t} = e^{aB_s - \frac{1}{2}a^2 s},$$

because for a Gaussian random variable X with mean 0 and variance σ^2, we have

$$\begin{aligned}
E(e^{aX}) &= \int_{-\infty}^{\infty} e^{ax} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2 - 2a\sigma^2 x}{\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x - a\sigma^2)^2 - a^2\sigma^4}{\sigma^2}} dx \\
&= e^{\frac{1}{2}a^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx = e^{\frac{1}{2}a^2\sigma^2}.
\end{aligned}$$

(*We have* $P(\sigma < \infty) = 1$:) By Doob's martingale convergence theorem, the martingale $B_{t\wedge\sigma}^2 - (t \wedge \sigma)$ converges to $B_\sigma^2 - \sigma$ almost surely. The reverse Fatou lemma gives

$$E(B_\sigma^2 - \sigma) \geq \limsup_{t\to\infty} E(B_{t\wedge\sigma}^2 - (t \wedge \sigma)) = 0$$

and hence $E(\sigma) \leq E(B_\sigma^2) < \infty$.

(*Compute* $P(\tau_a < \tau_b)$:) Like in the previous exercise,

$$E(M_\sigma) = E(M_0) = 0 = aP(\tau_a < \tau_b) + b(1 - P(\tau_a < \tau_b)),$$

so $P(\tau_a < \tau_b) = \frac{b}{b-a}$.