Stochastic analysis, 2. exercises

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Exercise 1 Let $0 < s < t < u$, and $(B_r : r \geq 0)$ a standard Brownian motion with $B_0 = 0$.

Compute the conditional distribution of B_t conditionally on $\sigma(B_s, B_u)$.

Solution 1 Let f_{B_s} be the probability density function of B_s,

$$f_s(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}. \tag{1}$$

Then the joint probability density of B_s, B_t and B_u is given by

$$\begin{aligned} f_{B_s,B_t,B_u}(x,y,z) &= f_s(x) f_{B_t|B_s=x}(y) f_{B_u|B_t=y}(z) \\ &= f_s(x) f_{t-s}(y-x) f_{u-t}(z-y) \\ &= \frac{1}{\sqrt{2\pi s}\sqrt{2\pi(t-s)}\sqrt{2\pi(u-t)}} e^{-\frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)} - \frac{(z-y)^2}{2(u-t)}}. \end{aligned} \tag{2}$$

The conditional probability density of B_t given $B_s = S$ and $B_u = U$ is

$$f_{B_t|B_s=S,B_u=U}(x) = \frac{f_{B_s,B_t,B_u}(S,x,U)}{\int_{-\infty}^{\infty} f_{B_s,B_t,B_u}(S,t,U)dt}. \tag{3}$$

By (2) we have

$$\begin{aligned} f_{B_s,B_t,B_u}(S,x,U) &= \frac{1}{\sqrt{2\pi s}\sqrt{2\pi(t-s)}\sqrt{2\pi(u-t)}} e^{-\frac{S^2}{2s} - \frac{(x-S)^2}{2(t-s)} - \frac{(U-x)^2}{2(u-t)}} \\ &= \frac{e^{-\frac{S^2}{2s} - \frac{(x^2-2Sx+S^2)(u-t)+(x^2-2Ux+U^2)(t-s)}{2(t-s)(u-t)}}}{\sqrt{2\pi s}\sqrt{2\pi(t-s)}\sqrt{2\pi(u-t)}} \\ &= \frac{e^{-\frac{S^2}{2s} - \frac{(u-s)x^2-2x(S(u-t)+U(t-s))+S^2(u-t)+U^2(t-s)}{2(t-s)(u-t)}}}{\sqrt{2\pi s}\sqrt{2\pi(t-s)}\sqrt{2\pi(u-t)}}, \end{aligned}$$

so

$$f_{B_t|B_s=S,B_u=U}(x) = \frac{e^{-\frac{(u-s)x^2-2x(S(u-t)+U(t-s))}{2(t-s)(u-t)}}}{\int_{-\infty}^{\infty} e^{-\frac{(u-s)y^2-2y(S(u-t)+U(t-s))}{2(t-s)(u-t)}} dy}.$$

By making the change of variables $y\sqrt{u-s} - \frac{S(u-t)+U(t-s)}{\sqrt{u-s}} = z$ we can compute

$$\int_{-\infty}^{\infty} e^{-\frac{(u-s)y^2-2y(S(u-t)+U(t-s))}{2(t-s)(u-t)}} dy = \int_{-\infty}^{\infty} e^{-\frac{z^2-\frac{(S(u-t)+U(t-s))^2}{u-s}}{2(t-s)(u-t)}} \frac{dz}{\sqrt{u-s}}$$

$$= \frac{e^{\frac{(S(u-t)+U(t-s))^2}{2(t-s)(u-t)(u-s)}}}{\sqrt{u-s}} \sqrt{2\pi(t-s)(u-t)},$$

and hence

$$f_{B_t|B_s=S,B_u=U}(x) = \frac{e^{-\frac{(u-s)x^2-2x(S(u-t)+U(t-s))+\frac{(S(u-t)+U(t-s))^2}{(u-s)}}{2(t-s)(u-t)}}}{\sqrt{2\pi\frac{(t-s)(u-t)}{u-s}}}$$

$$= \frac{1}{\sqrt{2\pi\frac{(t-s)(u-t)}{u-s}}} e^{-\frac{\left(x-\frac{S(u-t)+U(t-s)}{u-s}\right)^2}{2\frac{(t-s)(u-t)}{u-s}}}.$$

Thus B_t given $B_s = S$, $B_u = U$ is normally distributed with mean $\frac{S(u-t)+U(t-s)}{u-s}$ and variance $\frac{(t-s)(u-t)}{u-s}$.

Exercise 2 Let $(a_t : t \in \mathbb{N})$, $(b_t : t \in \mathbb{N})$ sequences. Denote $\Delta a_t = a_t - a_{t-1}$, $\Delta b_t = b_t - b_{t-1}$.

Check Abel's discrete integration by parts formula:

$$a_T b_T = a_0 b_0 + \sum_{t=1}^{T} a_{t-1}\Delta b_t + \sum_{t=1}^{T} b_t \Delta a_t$$
$$= a_0 b_0 + \sum_{t=1}^{T} a_t \Delta b_t + \sum_{t=1}^{T} b_{t-1}\Delta a_t$$
$$= a_0 b_0 + \sum_{t=1}^{T} a_{t-1}\Delta b_t + \sum_{t=1}^{T} b_{t-1}\Delta a_t + \sum_{t=1}^{T} \Delta a_t \Delta b_t.$$

Solution 2 We have

$$a_0 b_0 + \sum_{t=1}^{T} a_{t-1}\Delta b_t + \sum_{t=1}^{T} b_t \Delta a_t$$
$$= a_0 b_0 + \sum_{t=1}^{T} \left(a_{t-1}(b_t - b_{t-1}) + b_t(a_t - a_{t-1})\right)$$
$$= a_0 b_0 + \sum_{t=1}^{T} (a_{t-1}b_t - a_{t-1}b_{t-1} + b_t a_t - b_t a_{t-1})$$
$$= a_0 b_0 - a_0 b_0 + a_T b_T + \sum_{t=1}^{T} (-a_t b_t + b_t a_t) = a_T b_T.$$

Similarly

$$a_T b_T = a_0 b_0 + \sum_{t=1}^{T} a_t \Delta b_t + \sum_{t=1}^{T} b_{t-1} \Delta a_t.$$

Finally

$$\begin{aligned} & a_0 b_0 + \sum_{t=1}^{T} a_{t-1} \Delta b_t + \sum_{t=1}^{T} b_{t-1} \Delta a_t + \sum_{t=1}^{T} \Delta a_t \Delta b_t \\ &= a_0 b_0 + \sum_{t=1}^{T} \Big(a_{t-1} b_t - a_{t-1} b_{t-1} + b_{t-1} a_t - b_{t-1} a_{t-1} \\ &\quad + a_t b_t - a_t b_{t-1} - a_{t-1} b_t + a_{t-1} b_{t-1} \Big) \\ &= a_0 b_0 + \sum_{t=1}^{T} (a_t b_t - a_{t-1} b_{t-1}) = a_T b_T. \end{aligned}$$

Exercise 3 Let x_t be a continuous path with quadratic variation $[x,x]_t$ among the dyadic sequence of partitions Π_n, and a_t a continuous process with finite variation.

Use Ito formula to show the integration by parts formula:

$$x_t a_t = x_0 a_0 + \int_0^t a_s dx_s + \int_0^t x_s da_s.$$

Solution 3 Let $F(x,a) = xa$. Then $F \in C^{2,1}$ and we can use the extended Ito formula

$$F(x_t, a_t) - F(x_0, a_0) - \int_0^t F_a(x_s, a_s) da_s - \frac{1}{2} \int_0^t F_{xx}(x_s, a_s) d[x,x]_s = \int_0^t F_x(x_s, a_s) dx_s, \quad (4)$$

from the 1. exercise set to get

$$x_t a_t = x_0 a_0 + \int_0^t a_s dx_s + \int_0^t x_s da_s.$$

Exercise 4 Let x_t be a continuous path with quadratic variation $[x,x]_t$ among the dyadic sequence of partitions Π_n, and $z_0 > 0$.

Show that $z_t = z_0 \exp(x_t - \frac{1}{2}[x,x]_t)$ satisfies the linear pathwise differential equation

$$dz_t = z_t dx_t,$$

which is understood in the integral sense

$$z_t = z_0 + \int_0^t z_s \overleftarrow{dx_s}.$$

Solution 4 Consider the function $F(x,a) = z_0 \exp(x - \frac{1}{2}a)$. It is clearly in $C^{2,1}$ and thus we can use the extended Ito formula (4) with $a_t = [x,x]_t$ to get

$$z_t = z_0 \exp(x_0 - \tfrac{1}{2}[x,x]_0) + \int\limits_0^t z_0 \exp(x_s - \tfrac{1}{2}[x,x]_s) \overleftarrow{dx_s}$$

$$- \tfrac{1}{2} \int\limits_0^t z_0 \exp(x_s - \tfrac{1}{2}[x,x]_s) d[x,x]_s$$

$$+ \tfrac{1}{2} \int\limits_0^t z_0 \exp(x - \tfrac{1}{2}[x,x]_s) d[x,x]_s = z_0 + \int\limits_0^t z_s \overleftarrow{dx_s} .$$

Exercise 5 What is the quadratic variation of z_t?

Solution 5 Clearly $x_t - \frac{1}{2}[x,x]_t$ has quadratic variation $[x,x]_t$. Now $z_t = f(x_t - \frac{1}{2}[x,x]_t)$, where $f(x) = z_0 \exp(x)$. Thus z_t has quadratic variation

$$[z,z]_t = \int\limits_0^t f'(x_s - \tfrac{1}{2}[x,x]_s)^2 d[x - \tfrac{1}{2}[x,x], x - \tfrac{1}{2}[x,x]]_s = \int\limits_0^t z_s^2 d[x,x]_s.$$

Exercise 6 Show that $z_t^{-1} = z_0^{-1} \exp(-x_t + \frac{1}{2}[x,x]_t)$ satisfies

$$z_t^{-1} = z_0^{-1} - \int\limits_0^t z_s^{-1} dx_s + \int\limits_0^1 z_s^{-1} d[x,x]_s.$$

Remarks: note that from the assumptions it follows that z_t is bounded away from zero on any compact interval, which means that $1/z_t$ is bounded on compacts.

Note that by definition $[x,x]_t = [-x,-x]_t$.

Solution 6 Letting $F(x,a) = z_0^{-1} \exp(-x + \frac{1}{2}a)$ and using the extended Ito formula (4) with $a_t = [x,x]_t$ we get

$$z_t^{-1} = z_0^{-1} - \int\limits_0^t z_0^{-1} \exp(-x_s + \tfrac{1}{2}[x,x]_s) \overleftarrow{dx_s}$$

$$+ \tfrac{1}{2} \int\limits_0^t z_0^{-1} \exp(-x_s + \tfrac{1}{2}[x,x]_s) d[x,x]_s$$

$$+ \tfrac{1}{2} \int\limits_0^t z_0^{-1} \exp(-x_s + \tfrac{1}{2}[x,x]_s) d[x,x]_s$$

$$= z_0^{-1} - \int\limits_0^t z_s^{-1} \overleftarrow{dx_s} + \int\limits_0^t z_s^{-1} d[x,x]_s$$

Exercise 7 Let a_t be a continuous path with finite first variation, and z_t as before. Show that

$$\xi_t = \left(1 + \int\limits_0^t \tfrac{1}{z_s} da_s \right) z_t$$

satisfies the linear inhomogeneous pathwise differential equation

$$d\xi_t = \xi_t dx_t + da_t, \quad \xi_0 = z_0.$$

Solution 7 Let $b_t = 1 + \int\limits_0^t \frac{1}{z_s} da_s$. Now by the integration by parts formula from 3. exercise we have

$$z_t b_t = z_0 b_0 + \int\limits_0^t b_s dz_s + \int\limits_0^t z_s db_s.$$

Clearly $db_s = \frac{1}{z_s} da_s$ and $dz_s = z_s dx_s$ by exercise 4. Thus

$$z_t b_t = z_0 b_0 + \int\limits_0^t b_s z_s dx_s + \int\limits_0^t da_s,$$

or

$$\xi_t = \xi_0 + \int\limits_0^t \xi_s dx_s + a_t - a_0,$$

implying that

$$d\xi_t = \xi_t dx_t + da_t.$$

Exercise 8 Let a_t be a continuous path with finite first variation and x_t continuous with quadratic variation $[x, x]_t$ among the dyadic sequence of partitions. Show that

$$\int\limits_0^t a_s dx_s = a_t x_t - a_0 x_0 - \int\limits_0^t x_s da_s = \lim_{\Delta(\Pi) \to 0} \sum_{t_i \in \Pi} a_{t_i} (x_{t_{i+1} \wedge t} - x_{t_i \wedge t})$$

is well-defined independently of the sequence of partitions.

Solution 8 By Abel's discrete integration by parts formula we have

$$\sum_{t_i \in \Pi} a_{t_i} (x_{t_{i+1} \wedge t} - x_{t_i \wedge t}) = a_t x_t - a_0 x_0 - \sum_{t_i \in \Pi} x_{t_i} (a_{t_i \wedge t} - a_{t_{i-1} \wedge t}).$$

Because a_t has finite first variation, the Riemann-Stieltjes integral exists, and

$$\lim_{\Delta(\Pi) \to 0} \sum_{t_i \in \Pi} x_{t_i} (a_{t_i \wedge} - a_{t_{i-1} \wedge t}) = \int\limits_0^t x_s da_s.$$

Exercise 9 Show that $y_t = \int\limits_0^t a_s dx_s$ has quadratic variation among the dyadic sequence partitions given by

$$[y, y]_t = \int\limits_0^t a_s^2 d[x, x]_s.$$

Solution 9 Let Π be a partition with largest element t. We have

$$\sum_{t_i \in \Pi} (y_{t_{i+1}} - y_{t_i})^2 = \sum_{t_i \in \Pi} \left(\int\limits_{t_i}^{t_{i+1}} a_s dx_s \right)^2.$$

By 8. exercise we have

$$\int_{t_i}^{t_{i+1}} a_s dx_s = a_{t_{i+1}} x_{t_{i+1}} - a_{t_i} x_{t_i} - \int_{t_i}^{t_{i+1}} x_s da_s = a_{t_i}(x_{t_{i+1}} - x_{t_i}) + \int_{t_i}^{t_{i+1}} (x_{t_{i+1}} - x_s) da_s,$$

so

$$\sum_{t_i \in \Pi} (y_{t_{i+1}} - y_{t_i})^2 = \sum_{t_i \in \Pi} \left(a_{t_i}^2 (x_{t_{i+1}} - x_{t_i})^2 + 2a_{t_i}(x_{t_{i+1}} - x_{t_i}) \int_{t_i}^{t_{i+1}} (x_{t_{i+1}} - x_{t_s}) da_s \right.$$
$$\left. + \left(\int_{t_i}^{t_{i+1}} (x_{t_{i+1}} - x_{t_s}) da_s \right)^2 \right).$$

Now as $\Delta(\Pi) \to 0$, we have $\sum_{t_i \in \Pi} a_{t_i}^2 (x_{t_{i+1}} - x_{t_i})^2 \to \int_0^t a_s^2 d[x,x]_s$. Moreover

$$\left| \int_{t_i}^{t_{i+1}} (x_{t_{i+1}} - x_{t_s}) da_s \right| \leq \int_{t_i}^{t_{i+1}} |x_{t_{i+1}} - x_{t_s}| dv_a(s) \leq \delta(t_{i+1} - t_i)(v_a(t_{i+1}) - v_a(t_i)),$$

where $\delta(\varepsilon)$ is such that $|x_u - x_v| < \delta(\varepsilon)$ whenever $0 \leq u, v \leq t$, $|u - v| \leq \varepsilon$. It follows that as $\Delta(\Pi) \to 0$ (so $\varepsilon \to 0$), the two other terms in the sum go to 0.