Stochastic analysis, 1. exercises

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Exercise 1 Let $x\colon \mathbb{R}^+ \to \mathbb{R}$ be a function. Its total variation (or first-variation) is defined as

$$v_t(x) = \sup_{\Pi} \sum_{t_i \in \Pi} |x_{t_i \wedge t} - x_{t_{i-1} \wedge t}|$$

where the supremum is over all finite partitions

$$\Pi = \{0 \le t_0 < t_1 < \ldots < t_n\},$$

$t \wedge s = \min(t, s)$.

Show that

$$v_t(x) = \lim_{\Delta(\Pi) \to 0} \sum_{t_i \in \Pi} |x_{t_i \wedge t} - x_{t_{i-1} \wedge t}|$$

where $\Delta(\Pi) = \max\{|t_i - t_{i-1}| : t_i \in \Pi\}$ and Π is a finite partition. This means that the limit exists for any sequence of partitions (Π_n) with $\Delta(\Pi_n) \to 0$ and the limiting value does not depend on the sequence.

Assume $v_T(x) < \infty$ for some $T \in (0, \infty]$. For $t \in [0, T]$, let

$$x_t^{\oplus} = \tfrac{v_t(x) + x_t - x_0}{2}, \quad x_t^{\ominus} = \tfrac{v_t(x) - x_t + x_0}{2}.$$

Show that $x_t^{\oplus}$ and $x_t^{\ominus}$ are non-decreasing satisfying $x_0^{\oplus} = x_0^{\ominus} = 0$ and

$$x_t = x_0 + x_t^{\oplus} - x_t^{\ominus}, v_t(x) = x_t^{\oplus} + x_t^{\ominus}. \tag{1}$$

Show that if

$$x_t = x_0 + y_t^{\oplus} - y_t^{\ominus} \tag{2}$$

with $y_t^{\oplus}$ and $y_t^{\ominus}$ non-decreasing satisfying $y_t^{\oplus} = y_t^{\ominus} = 0$, then

$$v_t(x) \le y_t^{\oplus} + y_t^{\ominus}.$$

Show that the decomposition (1) is minimal among decompositions (2), meaning that

$$a_t := y_t^{\oplus} - x_t^{\oplus} = y_t^{\ominus} - x_t^{\ominus}$$

is non-decreasing.

Solution 1 Suppose that x_t is continuous. We wish to show that

$$\sum_{t_i \in \Pi_n} |x_{t_i} - x_{t_{i-1}}| \to \sup_{\Pi} \sum_{t_i \in \Pi} |x_{t_i} - x_{t_{i-1}}|,$$

where Π_n and Π are partitions of $[0,t]$ such that $\Delta(\Pi_n) \to 0$. To obtain a contradiction, assume that there exists a partition Π and $\varepsilon > 0$ such that

$$\sum_{t_i \in \Pi_n} |x_{t_i} - x_{t_{i-1}}| < \sum_{s_i \in \Pi} |x_{s_i} - x_{s_{i-1}}| - \varepsilon$$

for all n. Now x_t is uniformly continuous on $[0,t]$, and hence there exists $\delta > 0$ such that $|x_a - x_b| < \frac{\varepsilon}{2N}$ when $|a-b| < \delta$, where N is the number of points in Π. For large enough n we then have

$$\begin{aligned}\sum_{t_i \in \Pi_n} |x_{t_i} - x_{t_{i-1}}| &= \sum_{s_i \in \Pi} \sum_{t_i \in \Pi_n, s_{i-1} < t_i \le s_i} |x_{t_i} - x_{t_{i-1}}| \\ &\ge \sum_{s_i \in \Pi} (|x_{s_i} - x_{s_{i-1}}| - 2\tfrac{\varepsilon}{2N}) = \sum_{s_i \in \Pi} |x_{s_i} - x_{s_{i-1}}| - \varepsilon,\end{aligned}$$

a contradiction.

Assuming that $v_T(x) < \infty$ for some $T \in (0, \infty]$, we have

$$x_0 + x_t^{\oplus} - x_t^{\ominus} = x_0 + \tfrac{v_t(x) + x_t - x_0 - v_t(x) + x_t - x_0}{2} = x_t$$

and similarly

$$x_t^{\oplus} + x_t^{\ominus} = \tfrac{v_t(x) + x_t - x_0 + v_t(x) - x_t + x_0}{2} = v_t(x).$$

Suppose then that $x_t = x_0 + y_t^{\oplus} - y_t^{\ominus}$ with $y_t^{\oplus}$ and $y_t^{\ominus}$ non-decreasing satisfying $y_t^{\oplus} = y_t^{\ominus} = 0$. We want to show that $x_t^{\oplus} - x_s^{\oplus} \le y_t^{\oplus} - y_s^{\oplus}$ and similarly $x_t^{\ominus} - x_s^{\ominus} \le y_t^{\ominus} - y_s^{\ominus}$ for $s < t$. Once this is done, it clearly follows that a_t is non-decreasing and $v_t(x) = x_t^{\oplus} + x_t^{\ominus} \le y_t^{\oplus} - y_s^{\oplus} + x_s^{\oplus} + y_t^{\ominus} - y_s^{\ominus} + x_s^{\ominus} \le y_t^{\oplus} + y_t^{\ominus}$ because $y_t^{\oplus} \ge x_t^{\oplus}$ for all t.

Notice that $v_t(x) = v_t(x_0 + y_t^{\oplus} - y_t^{\ominus}) = v_t(y_t^{\oplus} - y_t^{\ominus})$ and hence

$$x_t^{\oplus} - x_s^{\oplus} = \tfrac{v_t(x) - v_s(x) + x_t - x_s}{2} = \tfrac{v_t(y^{\oplus} - y^{\ominus}) - v_s(y^{\oplus} - y^{\ominus}) + y_t^{\oplus} - y_t^{\ominus} - y_s^{\oplus} + y_s^{\ominus}}{2}.$$

Now

$$\begin{aligned} v_t(y^\oplus - y^\ominus) - v_s(y^\oplus - y^\ominus) &= \sup_\Pi \sum_{s\le t_i\le t, t_i\in\Pi} |y^\oplus_{t_i} - y^\oplus_{t_{i-1}} - y^\ominus_{t_i} + y^\ominus_{t_{i-1}}| \\ &\le \sup_\Pi \sum_{s\le t_i\le t, t_i\in\Pi} (y^\oplus_{t_i} - y^\oplus_{t_{i-1}} + y^\ominus_{t_i} - y^\ominus_{t_{i-1}}) \\ &= y^\oplus_t - y^\oplus_s + y^\ominus_t - y^\ominus_s, \end{aligned}$$

so

$$x^\oplus_t - x^\oplus_s \le y^\oplus_t - y^\oplus_s.$$

The other inequality is shown similarly.

Exercise 2 Assume that x_t is a continuous path with quadratic variation $[x]_t$ among the sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$, and let a_t be a continuous function with finite first variation on compact intervals, that is $v_t(a) < \infty$. Let $F(x,a)$ be a $C^{2,1}$-function, with continuous partial derivatives $(x,a) \to F_{xx}(x,a)$ and $(x,a) \to F_a(x,a)$.

Use Taylor expansion, uniform continuity, and the weak convergence definition of the quadratic variation to show in details the extended Ito-Föllmer formula

$$F(x_t,a_t) - F(x_0,a_0) - \int_0^t F_a(x_s,a_s)da_s - \frac{1}{2}\int_0^t F_{xx}(x_s,a_s)d[x]_s =$$

$$\int_0^t F_x(x_s,a_s)d\overleftarrow{x}_s = \lim_{n\to\infty} \sum_{t_i^n\in\Pi_n} F_x(x_{t_i^n}, a_{t_i^n})(x_{t_{i+1}^n\wedge t} - x_{t_i^n\wedge t})$$

where the last equality defines the pathwise integral.

Solution 2 We have

$$\begin{aligned} F(x_t,a_t) - F(x_0,a_0) &= \lim_{n\to\infty} \sum_{t_i^n\in\Pi_n} (F(x_{t_{i+1}^n}, a_{t_{i+1}^n}) - F(x_{t_i^n}, a_{t_i^n})) \\ &= \lim_{n\to\infty} \sum_{t_i^n\in\Pi_n} (F(x_{t_{i+1}^n}, a_{t_{i+1}^n}) - F(x_{t_i^n}, a_{t_{i+1}^n}) + F(x_{t_i^n}, a_{t_{i+1}^n}) - F(x_{t_i^n}, a_{t_i^n})) \\ &= \lim_{n\to\infty} \sum_{t_i^n\in\Pi_n} \big(F_a(x_{t_{i+1}^n}, a_{\tau_i^n})(a_{t_{i+1}^n} - a_{t_i^n}) + F_x(x_{t_i^n}, a_{t_i^n})(x_{t_{i+1}^n} - x_{t_i^n}) \\ &\quad + \tfrac{1}{2}F_{xx}(x^n_{t_i}, a^n_{t_i})(x_{t_{i+1}^n} - x_{t_i^n})^2 + \tfrac{1}{2}(F_{xx}(x_{\sigma_i^n}, a_{t_i^n}) - F_{xx}(x_{t_i^n}, a_{t_i^n}))(x_{t_{i+1}^n} - x_{t_i^n})^2\big) \end{aligned}$$

$$= \int_0^t F_x(x_s,a_s)d\overleftarrow{x}_s + \int_0^t F_a(x_s,a_s)da_s + \frac{1}{2}\int_0^t F_{xx}(x_s,a_s)d[x]_s$$
$$+ \frac{1}{2}\lim_{n\to\infty} \sum_{t_i^n\in\Pi_n} (F_{xx}(x_{\sigma_i^n},a_{t_i^n}) - F_{xx}(x_{t_i^n},a_{t_i^n}))(x_{t_{i+1}^n} - x_{t_i^n})^2.$$

By uniform continuity we can find a sequence $C_n \to 0$ as $n \to \infty$, such that $F_{xx}(x_{\sigma_i^n},a_{t_i^n}) - F_{xx}(x_{t_i^n},a_{t_i^n}) \le C_n$ for all $n \in \mathbb{N}$ and hence

$$\sum_{t_i^n\in\Pi_n} (F_{xx}(x_{\sigma_i^n},a_{t_i^n}) - F_{xx}(x_{t_i^n},a_{t_i^n}))(x_{t_{i+1}^n} - x_{t_i^n})^2 \le C_n[x]_n \to 0$$

as $n \to \infty$.

Exercise 3 Assume that x_t is a continuous path with $x_0 = 0$, and quadratic variation $[x]_t = t$, among the dyadic sequence of partitions $\mathcal{D} = (t_k^n = k2^{-n} : k \in \mathbb{N})_{n\in\mathbb{N}}$, and let $a_t = \exp(t)$. Use the change of variables formula of classical Riemann-Stieltjes integrals and Ito-Föllmer formula (??) to compute the integral representation of

- $\sin(a_t)$,
- $\sin(x_t)$,
- $\sin(a_t x_t)$.

Solution 3 We have

$$\sin(a_t) = \sin(a_0) + \int_0^t \cos(a_s)da_s + \frac{1}{2}\int_0^t (-\sin(a_s))d[a]_s$$
$$= \sin(1) + \int_0^t \cos(a_s)da_s,$$

since $[a]_s = 0$.

Also

$$\sin(x_t) = \sin(x_0)+\int_0^t \cos(x_s)dx_s+\frac{1}{2}\int_0^t (-\sin(x_s))d[x]_s = \int_0^t \cos(x_s)dx_s+\frac{1}{2}\int_0^t (-\sin(x_s))ds$$

and if we let $F(x,a) = \sin(xa)$, then $F_x(x_s,a_s) = a_s\cos(a_s x_s)$, $F_a(x_s,a_s) = x_s\cos(a_s x_s)$ and $F_{xx}(x_s,a_s) = -a_s^2\sin(x_s a_s)$, so by (??)

$$\sin(a_t x_t) = \int_0^t F_x(x_s, a_s) d\overleftarrow{x}_s + \int_0^t F_a(x_s, a_s) da_s + \frac{1}{2}\int_0^t F_{xx}(x_s, a_s) d[x]_s$$
$$= \int_0^t a\cos(a_s x_s) d\overleftarrow{x}_s + \int_0^t x_s \cos(a_s x_s) da_s - \frac{1}{2}\int_0^t a_s^2 \sin(x_s a_s) ds.$$

Exercise 4 What is the first variation of $\sin(a_t)$?

What is the quadratic variation of $\sin(x_t)$?

What is the quadratic variation of $\sin(a_t x_t)$?

Show that $\sin(x_t)$ and $\sin(a_t x_t)$ have infinite first variation.

Solution 4 The first variation of $f(t) = \sin(a_t)$ is given by

$$\int_0^t |f'(s)| ds.$$

We have $f'(s) = \cos(e^s)e^s$, and $f'(s) \geq 0$ if and only if $\cos(e^s) \geq 0$, or $s \in \log[-\pi/2 + 2k\pi, \pi/2 + 2k\pi]$ for some $k \in \mathbb{N}$. Thus

$$\int_0^t |f'(s)| ds = A + \sum_{k=1}^{M} \left(\int_{\log(2k\pi-\pi/2)}^{\log(2k\pi+\pi/2)} f'(s) ds - \int_{\log(2k\pi+\pi/2)}^{\log(2k\pi+3\pi/2)} f'(s) ds \right) + B$$
$$= A + B + \sum_{k=1}^{M} 4 = A + B + 4M,$$

where

$$A = \int_0^{\log(3\pi/2)} |f'(s)| ds = \int_0^{\log(\pi/2)} f'(s) ds - \int_{\log(\pi/2)}^{\log(3\pi/2)} f'(s) ds = 3 - sin(1),$$
$$M = \lfloor \frac{e^t}{2\pi} - 1/4 \rfloor$$

and

$$B = \int_{\log(2M\pi+3\pi/2)}^{t} |f'(s)| ds.$$

Consider next the quadratic variation of $\sin(x_t)$ along the sequence of Dyadic partitions. It is

$$\int_0^t \cos^2(x_s)d[x]_s = \int_0^t \cos^2(x_s)ds.$$

Next we notice that $a_t x_t$ has quadratic variation $[ax]_t = \int_0^t a_s^2 d[x]_s = \int_0^t e^{2s}ds = \frac{e^{2t}-1}{2}$. This follows because

$$\begin{aligned}\sum(a_{t_{i+1}}x_{t_{i+1}} - a_{t_i}x_{t_i})^2 &= \sum((a_{t_{i+1}} - a_{t_i})x_{t_{i+1}} + a_{t_i}(x_{t_{i+1}} - x_{t_i}))^2 \\ &= \sum(a_{t_{i+1}} - a_{t_i})^2 x_{t_{i+1}}^2 + 2a_{t_i}x_{t_{i+1}}(a_{t_{i+1}} - a_{t_i})(x_{t_{i+1}} - x_{t_i}) + a_{t_i}^2(x_{t_{i+1}} - x_{t_i})^2 \\ &\to \int_0^t a_s^2 d[x]_s.\end{aligned}$$

Hence $\sin(a_t x_t)$ has quadratic variation

$$\int_0^t \cos^2(a_s x_s)d[ax]_s = \int_0^t \cos^2(a_s x_s)e^{2s}ds.$$

Neither $\sin(x_t)$ or $\sin(a_t x_t)$ have finite first variation, because their quadratic variations are strictly positive.