Stochastic analysis, 11. exercises

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Exercise 1 Let $X_t(\omega) \geq 0$ with $X_0 = 0$, and $A_t(\omega) \geq 0$ be continuous processes adapted with respect to $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, and assume that A_t is non-decreasing such that for all **bounded** stopping times $\tau(\omega)$

$$E(X_\tau) \leq E(A_\tau) \tag{1}$$

We introduce the running maximum $X_t^*(\omega) = \max_{0 \leq s \leq t} X_s(\omega)$.

Prove the following inequalities for **all** $\mathbb{F}$-stopping times τ (also unbounded): $\forall \varepsilon, \delta > 0$

a) $P(X_\tau^* > \varepsilon) \leq \frac{E(A_\tau)}{\varepsilon}$

b) $P(X_\tau^* > \varepsilon, A_\tau \leq \delta) \leq \frac{E(A_\tau \wedge \delta)}{\varepsilon}$

c) $P(X_\tau^* > \varepsilon) \leq \frac{E(A_\tau \wedge \delta)}{\varepsilon} + P(A_\tau > \delta)$

Solution 1 Let $\varepsilon, \delta > 0$ and define

$$\sigma(\omega) = \inf\{t : X_t(\omega) > \varepsilon\}.$$

Then $\{X_\tau^* > \varepsilon\} = \{\sigma < \tau\}$. Assume that τ is a bounded stopping time. Then (1) implies that

$$E(X_{\tau \wedge \sigma}) \leq E(A_{\tau \wedge \sigma}).$$

Now we can write

$$E(X_{\tau \wedge \sigma}) = E(X_{\tau \wedge \sigma}; \{\sigma < \tau\}) + E(X_{\tau \wedge \sigma}; \{\sigma \geq \tau\}) = \varepsilon P(X_\tau^* > \varepsilon) + E(X_\tau; \{\sigma \geq \tau\}).$$

Thus we have

$$\varepsilon P(X_\tau^* > \varepsilon) \leq E(A_{\tau \wedge \sigma}) - E(X_\tau; \{\sigma \geq \tau\}).$$

Because A_t is non-decreasing and $X_t \geq 0$, this implies (a). We note that (b) implies (c) since

$$P(X_\tau^* > \varepsilon) = P(X_\tau^* > \varepsilon, A_\tau \leq \delta) + P(X_\tau^* > \varepsilon, A_\tau > \delta) \leq \frac{E(A_\tau \wedge \delta)}{\varepsilon} + P(A_\tau > \delta).$$

To show (b), we define $\eta(\omega) = \inf\{t : A_t(\omega) > \delta\}$. Then

$$E(X_{\tau \wedge \sigma \wedge \eta}) \leq E(A_{\tau \wedge \sigma \wedge \eta}) \leq E(A_\tau \wedge \delta).$$

Now

$$\begin{aligned} E(X_{\tau\wedge\sigma\wedge\eta}) &= E(X_{\tau\wedge\sigma\wedge\eta}; \sigma < \tau) + E(X_{\tau\wedge\sigma\wedge\eta}; \sigma \geq \tau) \\ &\geq E(X_{\sigma\wedge\eta}; X^*_\tau > \varepsilon, \sigma < \eta) + E(X_{\sigma\wedge\eta}; X^*_\tau > \varepsilon, \sigma \geq \eta) \\ &\geq E(\varepsilon; X^*_\tau > \varepsilon, A_\tau \leq \delta) \end{aligned}$$

and the result follows.

Assume then that τ is not necessarily bounded. Then there exists a sequence τ_n of bounded stopping times, $\tau_n \uparrow \tau$. Notice that X^*_t and A_t are increasing processes. Therefore it follows from monotone convergence theorem and dominated convergence (In the case of $P(X^*_t > \varepsilon, A_\tau \leq \delta)$) that (a), (b) and (c) hold for general τ.

Exercise 2 Let M_t be a continuous $\mathbb{F}$-local martingale. The $\mathbb{F}$-predictable variation $\langle M\rangle_t$ is the non-decreasing process with $\langle M\rangle_0 = 0$ such that

$$N_t = M_t^2 - \langle M\rangle_t$$

is a local $\mathbb{F}$-martingale.

Show that for any $\mathbb{F}$-stopping time τ

$$P\left(\max_{0\leq s\leq\tau(\omega)} |M_s(\omega)| > \varepsilon\right) \leq \frac{E(\delta \wedge \langle M\rangle_t)}{\varepsilon^2} + P(\langle M\rangle_\tau > \delta).$$

Solution 2 By the first exercise, it is enough to show that

$$E(M_\tau^2) \leq E(\langle M\rangle_\tau)$$

for all bounded stopping times τ.

Let σ_n, $n \in \mathbb{N}$, be a localizing sequence for N_t. Then

$$E(\langle M\rangle_{\tau\wedge\sigma_n}) = E(M^2_{\tau\wedge\sigma_n} - N_{\tau\wedge\sigma_n}) = E(M^2_{\tau\wedge\sigma_n}).$$

By Fatou's lemma,

$$E(M_\tau^2) = E(\liminf_{n\to\infty} M^2_{\tau\wedge\sigma_n}) \leq \liminf_{n\to\infty} E(M^2_{\tau\wedge\sigma_n}) = \liminf_{n\to\infty} E(\langle M\rangle_{\tau\wedge\sigma_n}) = E(\langle M\rangle_\tau).$$

Exercise 3 Let $\{M_t^{(n)}(\omega)\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{F}$-local martingales and τ an $\mathbb{F}$-stopping time. Show that as $n \to \infty$

$$\langle M^{(n)}\rangle_\tau \to 0 \Rightarrow \max_{0\leq s\leq\tau} |M_s^{(n)}(\omega)| \to 0$$

with both convergences in probability.

Solution 3 Assume that $\langle M^{(n)}\rangle_\tau \to 0$ in probability as $n \to \infty$. Let $\varepsilon > 0$ and $\eta > 0$. We wish to show that for large enough n,

$$P\left(\max_{0\leq s\leq\tau} |M_s^{(n)}| > \varepsilon\right) \leq \eta.$$

Let $\delta = \frac{1}{2}\varepsilon^2\eta$. Then by the second exercise,

$$P\left(\max_{0\le s\le\tau}|M_s^{(n)}|>\varepsilon\right)\le\frac{\delta}{\varepsilon^2}+P(\langle M^{(n)}\rangle_\tau>\delta)\le\frac{1}{2}\eta+\frac{1}{2}\eta$$

for n large enough since $\langle M^{(n)}\rangle_\tau\to 0$.

Exercise 4 Let $(B_t:t\ge 0)$ be a Brownian motion in the filtration $\mathbb{F}=\{\mathcal{F}_t:t\ge 0\}$.

(a) Use Ito formula to show that

$$Z_t(\theta)=\exp\left(\theta B_t-\frac{\theta^2}{2}t\right)$$

is a true martingale.

(b) Use Ito's formula to compute the semimartingale decomposition of B_t^n for $n\in\mathbb{N}$ into a local martingale and a process of finite variation. Show that the local martingale is a true martingale. Recall that a Gaussian random variable is in $L^p(\Omega)$ for all $p<\infty$.

(c) Compute $E(B_t^n)$ for $n\in\mathbb{N}$ by taking expectation in the semimartingale decomposition.

Solution 4 (a) Let $f(x)=e^x$ and $X_t=\theta B_t-\frac{\theta^2}{2}t$. Then $\langle X\rangle_t=\theta^2 t$ and by using the Ito formula we have

$$\begin{aligned}f(X_t)&=\int_0^t f'(X_s)\,dX_s+\frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s\\&=\int_0^t \theta e^{\theta B_s-\frac{\theta^2}{2}s}\,dB_s-\frac{\theta^2}{2}\int_0^t e^{\theta B_s-\frac{\theta^2}{2}s}\,ds+\frac{\theta^2}{2}\int_0^t e^{\theta B_s-\frac{\theta^2}{2}s}\,ds=\int_0^t \theta e^{\theta B_s-\frac{\theta^2}{2}s}\,dB_s.\end{aligned}$$

Because B_s is an L^2 martingale, it is enough to show that

$$E\left(\int_0^t e^{2\theta B_s-\theta^2 s}\,ds\right)<\infty.$$

Now

$$\begin{aligned}E\left(\int_0^t e^{2\theta B_s-\theta^2 s}\,ds\right)&=\int_0^t E(e^{2\theta B_s-\theta^2 s})\,ds\\&=\int_0^t e^{-\theta^2 s}\int_{-\infty}^{\infty} e^{2\theta x}\frac{1}{\sqrt{2\pi s}}e^{-\frac{x^2}{2s}}\,dx\,ds\\&=\int_0^t e^{-\theta^2 s}e^{2s\theta^2}\,ds=\frac{e^{t\theta^2}-1}{\theta^2}<\infty.\end{aligned}$$

(b) Assume that $n\ge 1$. Let $f(x)=x^n$. Using Ito formula we have

$$B_t^n=f(B_t)=\int_0^t f'(B_s)\,dB_s+\frac{1}{2}\int_0^t f''(B_s)\,d\langle B\rangle_s=n\int_0^t B_s^{n-1}\,dB_s+\frac{n(n-1)}{2}\int_0^t B_s^{n-2}\,ds.$$

Again it is enough to show that

$$E\left(\int_0^t B_s^{2(n-1)}\,ds\right) < \infty.$$

We have

$$\begin{aligned} E\left(\int_0^t B_s^{2(n-1)}\,ds\right) < \infty &= \int_0^t E(B_s^{2(n-1)})\,ds \\ &= \int_0^t (2n-3)!!s^{2n-2}\,ds = (2n-3)!!\frac{t^{2n-1}}{2n-1} < \infty. \end{aligned}$$

(c) Since the expectation of the martingale term in the semimartingale decomposition is 0, we have

$$E(B_t^n) = \frac{n(n-1)}{2}E\left(\int_0^t B_s^{n-2}\,ds\right) = \frac{n(n-1)}{2}\int_0^t E(B_s^{n-2})\,ds = \frac{n(n-1)}{2}\int_0^t (n-3)!!s^{n-2}\,ds = \frac{n(n-3)!!}{2}t^{n-1}.$$

Exercise 5 The Hermite polynomials are defined by the Taylor expansion

$$F(x,u) = \exp(ux - \frac{u^2}{2}) = \sum_{n=0}^{\infty}\frac{u^n}{n!}h_n(x). \tag{2}$$

We see that $h_0(x) = 1$. We rewrite

$$F(x,u) = \exp\left(\frac{x^2}{2} - \frac{(x-u)^2}{2}\right) = e^{x^2/2}\sum_{n=0}^{\infty}\frac{u^n}{n!}\frac{\partial^n}{\partial u^n}\exp\left(-\frac{(t-u)^2}{2}\right)\Big|_{u=0}$$

which shows that

$$h_n(x) = (-1)^n\exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2}\right), \quad n \geq 0.$$

We also have

$$\begin{aligned} F(\frac{x}{\sqrt{a}}, u\sqrt{a}) &= \exp\left(u\sqrt{a}\frac{x}{\sqrt{a}} - \frac{u^2a}{2}\right) = \exp\left(ux - \frac{u^2a}{2}\right) \\ &= \sum_{n=0}^{\infty}\frac{u^n}{n!}a^{n/2}h_n\left(\frac{x}{\sqrt{a}}\right) = \sum_{n=0}^{\infty}\frac{u^n}{n!}H_n(x,a) \end{aligned}$$

with

$$H_n(x,a) := a^{n/2}h_n\left(\frac{x}{\sqrt{a}}\right).$$

We also set $H_n(x,0) = x^n$.

(a) Show that

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}H_n(x,a) + \frac{\partial}{\partial a}H_n(x,a) = 0.$$

(b) Use Ito formula to show when B_t is an $\mathbb{F}$-Brownian motion,

$$H_n(B_t, t) = t^{n/2} h_n(B_t/\sqrt{t})$$

is a martingale. Justify the martingale property of the Ito integral.

(c) Use Ito formula to show when M_t is a continuous local martingale in the $\mathbb{F}$-filtration,

$$H_n(M_t, \langle M \rangle_t) = \langle M \rangle_t^{n/2} h_n\left(\frac{M_t}{\sqrt{\langle M \rangle_t}}\right)$$

is a local martingale.

(d) Show also that

$$H_n(M_t, \langle M \rangle_t) = n! \int_0^t \left(\int_0^{t_1} \dots \int_0^{t_{n-1}} dM_{t_n} dM_{t_{n-1}} \dots \right) dM_{t_1}$$

where on the right we have an **iterated Ito integral**.

Solution 5 (a) Consider the series

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(x,a) + \frac{\partial}{\partial a} H_n(x,a) \right) = \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial a} \right) \exp\left(ux - \frac{u^2 a}{2} \right) = 0.$$

(b) It follows from (c) that $H_n(B_t, t)$ is a local martingale with

$$H_n(B_t, t) = \int_0^t s^{\frac{n-1}{2}} h_n{}' \left(\frac{B_s}{\sqrt{s}} \right) dB_s.$$

Thus it remains to show that

$$E\left(\int_0^t s^{n-1} h_n{}' \left(\frac{B_s}{\sqrt{s}} \right)^2 ds \right) < \infty.$$

Because h_n is a polynomial, by using Fubini we get an integral of the form

$$\int_0^t s^{n-1} Q(s^{1/2})\, ds < \infty.$$

Here $Q(s)$ is a polynomial resulting from the Gaussian moments.

(c) We have

$$\begin{aligned}
\frac{\partial}{\partial x} H_n(x,a) &= a^{\frac{n-1}{2}} h_n{}' \left(\frac{x}{\sqrt{a}} \right) \\
\frac{\partial}{\partial a} H_n(x,a) &= -\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(x,a) = -\frac{1}{2} a^{\frac{n-2}{2}} h_n{}'' \left(\frac{x}{\sqrt{a}} \right) \\
\frac{\partial^2}{\partial x^2} H_n(x,a) &= a^{\frac{n-2}{2}} h_n{}'' \left(\frac{x}{\sqrt{a}} \right).
\end{aligned}$$

Thus by Ito formula,

$$H_n(M_t, \langle M\rangle_t) = \int_0^t \langle M\rangle_s^{\frac{n-1}{2}} h_n'\left(\frac{M_s}{\sqrt{\langle M\rangle_s}}\right) dM_s + \int_0^t -\frac{1}{2}\langle M\rangle_s^{\frac{n-2}{2}} h_n''\left(\frac{M_s}{\sqrt{\langle M\rangle_s}}\right) d\langle M\rangle_t$$
$$+ \frac{1}{2}\int_0^t \langle M\rangle_s^{\frac{n-2}{2}} h_n''\left(\frac{M_s}{\sqrt{\langle M\rangle_s}}\right) d\langle M\rangle_t$$
$$= \int_0^t \langle M\rangle_s^{\frac{n-1}{2}} h_n'\left(\frac{M_s}{\sqrt{\langle M\rangle_s}}\right) dM_s.$$

It follows that $H_n(M_t, \langle M\rangle_t)$ is a local martingale.

(d) We'll prove this by induction. The base case $n = 1$ is clear. Now let $n \in \mathbb{N}$ and assume that the claim holds for $n-1$. Then by induction

$$n! \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dM_{t_n} dM_{t_{n-1}} \dots dM_{t_1} = n \int_0^t H_{n-1}(M_{t_1}, \langle M\rangle_{t_1}) dM_{t_1}$$
$$= n \int_0^t \langle M\rangle_{t_1}^{\frac{n-1}{2}} h_{n-1}\left(\frac{M_{t_1}}{\sqrt{\langle M\rangle_{t_1}}}\right) dM_{t_1}$$
$$= \int_0^t \langle M\rangle_{t_1}^{\frac{n-1}{2}} h_n'\left(\frac{M_{t-1}}{\sqrt{\langle M\rangle_{t_1}}}\right) dM_{t_1}$$
$$= H_n(M_t, \langle M\rangle_t).$$

Here we have used the fact that $h_n'(x) = nh_{n-1}(x)$, which is immediate if one differentiates (2).