Stochastic analysis, 3. exercises

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Exercise 1 Show that

$$\sum_{n=0}^{\infty} P\left(\sup_{s,t\in[0,1]} |\Gamma_n(s,t)| > n^{-2}\right) < \infty$$

where $\Gamma_n(s,t) = \sum_{(d,d')\in E_n^{(2)}} \eta_{d,d'}(s,t)\xi_{d,d'}$.

Solution 1 Because $\Gamma_n(s,t)$ is a piecewise linear function, the maximum is attained at one of the points in $E_n^{(2)}$. Let $(s,t) \in E_n^{(2)}$. Then

$$\begin{aligned}
\Gamma_n(s,t) &= \sum_{(d,d')\in E_n^{(2)}} \eta_d(s)\eta_{d'}(t)\xi_{d,d'} \\
&= \sum_{d,d'\in E_n} \eta_d(s)\eta_{d'}(t)\xi_{d,d'} + \sum_{k=0}^{n-1}\sum_{d\in E_n}\sum_{d'\in E_k} \left(\eta_d(s)\eta_{d'}(t)\xi_{d,d'} + \eta_{d'}(s)\eta_d(t)\xi_{d',d}\right) \\
&= \eta_{d_n}(s)\eta_{d'_n}(t)\xi_{d_n,d'_n} + \sum_{k=0}^{n-1}\left(\eta_{d_n}(s)\eta_{d'_k}(t)\xi_{d_n,d'_k} + \eta_{d_k}(s)\eta_{d'_n}(t)\xi_{d_k,d'_n}\right).
\end{aligned}$$

Here $d_k \in E_k$ is the unique point at which $\eta_{d_k}(s) \neq 0$, if it exists, and any point in E_k (say 2^{-k}) otherwise. Define similarly $d'_k \in E_k$ to be the point at which $\eta_{d_k}(t) \neq 0$. Then we have

$$\Gamma_n(s,t) = \sum_{k=0}^{n} \eta_{d_n}(s)\eta_{d'_k}(t)\xi_{d_n,d'_k} + \sum_{k=0}^{n-1} \eta_{d_k}(s)\eta_{d'_n}(t)\xi_{d_k,d'_n}.$$

This is a Gaussian random variable with mean 0 and variance

$$\sum_{k=0}^{n} \eta_{d_n}(s)^2\eta_{d'_k}(t)^2 + \sum_{k=0}^{n-1} \eta_{d_k}(s)^2\eta_{d'_n}(t)^2.$$

In particular the variance is bounded by

$$\begin{aligned}
\sum_{k=0}^{n} 2^{-(n+1)}2^{-(k+1)} + \sum_{k=0}^{n-1} 2^{-(k+1)}2^{-(n+1)} &= 2^{-(n+2)}\tfrac{1-2^{-(n+1)}}{1-2^{-1}} + 2^{-(n+2)}\tfrac{1-2^{-n}}{1-2^{-1}} \\
&= 2^{-(n+2)}(2-2^{-n}) + 2^{-(n+2)}(2-2^{-n+1}) \\
&\leq 2^{-n}(1-2^{-n-2}-2^{-n-1}) \leq 2^{-n}.
\end{aligned}$$

We know that for a gaussian random variable X with mean 0 and variance σ^2,

$$P(|X| > \varepsilon) = 2\int_{\varepsilon}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx \le 2\int_{\varepsilon}^{\infty} \frac{x}{\varepsilon}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx = \sigma\frac{\sqrt{2}}{\varepsilon\sqrt{\pi}} e^{-\frac{1}{2}\frac{\varepsilon^2}{\sigma^2}},$$

so $P(|\Gamma_n(s,t)| > n^{-2}) \le 2^{-n/2} n^2 \sqrt{\frac{2}{\pi}} e^{-\frac{2^{-n+1}}{n^4}}$, which implies that

$$\begin{aligned} P(\sup_{s,t} |\Gamma_n(s,t)| > n^{-2}) &= P(\sup_{s,t \in E_n^{(2)}} |\Gamma_n(s,t)| > n^{-2}) \\ &\le \sum_{(s,t)\in E_n^{(2)}} P(|\Gamma_n(s,t)| > n^{-2}) \le 2^{2n-n/2} n^2 \sqrt{\frac{2}{\pi}} e^{-\frac{2^{-n+1}}{n^4}}. \end{aligned}$$

It follows that the sum

$$\sum_{n=0}^{\infty} P(\sup_{s,t} |\Gamma_n(s,t)| > n^{-2})$$

converges.

Exercise 2 Show that the limiting process has the postulated covariance structure.

Solution 2 We have to show that $\mathrm{Cov}(B(s,t), B(u,v)) = (s \wedge u)(t \wedge v)$. We have

$$\begin{aligned} E(B(s,t)B(u,v)) &= E\left(\left(\sum_{d\in D}\sum_{d'\in D} \eta_{d,d'}(s,t)\xi_{d,d'}\right)\left(\sum_{e\in D}\sum_{e'\in D} \eta_{e,e'}(u,v)\xi_{e,e'}\right)\right) \\ &= \sum_{d\in D}\sum_{d'\in D}\sum_{e\in D}\sum_{e'\in D} \eta_{d,d'}(s,t)\eta_{e,e'}(u,v)E(\xi_{d,d'}\xi_{e,e'}) \\ &= \sum_{d\in D}\sum_{d'\in D} \eta_{d,d'}(s,t)\eta_{d,d'}(u,v)E(\xi_{d,d'}^2) \\ &= \sum_{d\in D}\sum_{d'\in D} \int_{[0,s]\times[0,t]} \ddot{\eta}_{d,d'}(x,y)dxdy \int_{[0,u]\times[0,v]} \ddot{\eta}_{d,d'}(x,y)dxdy \\ &= \sum_{d\in D}\sum_{d'\in D} \langle \ddot{\eta}_{d,d'}, \chi_{[0,s]\times[0,t]}\rangle \langle \ddot{\eta}_{d,d'}, \chi_{[0,u]\times[0,v]}\rangle \\ &= \langle \chi_{[0,s]\times[0,t]}, \chi_{[0,u]\times[0,v]}\rangle \\ &= |([0,s]\times[0,t]) \cap ([0,u]\times[0,v])| = (s\wedge u)(t \wedge v). \end{aligned}$$

Exercise 3 In fact the 2-dimensional Brownian sheet has also the α-Hölder continuity property with probability one. For which $\alpha > 0$?

Solution 3 Let $s,t,u,v \in [0,1]$ and consider the increment $B(s,t,\omega) - B(u,v,\omega) = X(\omega)$. We know that this is a Gaussian random variable with mean 0 and variance

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{Cov}(X,X) \\ &= \mathrm{Var}(B(s,t)) + \mathrm{Var}(B(u,v)) - 2\,\mathrm{Cov}(B(s,t), B(u,v)) \\ &= st + uv - 2(s\wedge u)(t\wedge v). \end{aligned}$$

Furthermore, if $p > 0$, then

$$
\begin{aligned}
E(|X|^p) &= \int_{-\infty}^{\infty} |x|^p \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx \\
&= \int_{-\infty}^{\infty} |y\sigma|^p \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \sigma^p \int_{-\infty}^{\infty} |y|^p \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \sigma^p E(|N|^p) \leq C\sigma^p.
\end{aligned}
$$

Here N is a random variable with standard Gaussian distribution, $E(|N|^p) \leq C$. Thus we have

$$E(|X|^p) = C(st + uv - 2(s \wedge u)(t \wedge v))^{p/2}.$$

Next we have to relate the variance $\sigma^2 = st + uv - 2(s \wedge u)(t \wedge v)$ to the distance $|(s,t) - (u,v)|$.

Lemma 1 Let $s, t, u, v \in [0,1]$. Then we have the inequality

$$st + uv - 2(s \wedge u)(t \wedge v) \leq |s-u| + |t-v| \left(\leq \sqrt{2}\sqrt{(s-u)^2 + (t-v)^2}\right). \tag{1}$$

Moreover

$$st + uv - 2(s \wedge u)(t \wedge v) \leq C((s-u)^2 + (t-v)^2)^p \tag{2}$$

does not hold for any $p > \frac{1}{2}$.

Proof. First notice that if $p > \frac{1}{2}$, then letting $s = 0$, $t = v = 1$ in (2) we have

$$u \leq Cu^{2p},$$

which cannot hold for u small enough.

To prove (1), we can consider a few cases. Suppose first that $s \leq u$, $t \leq v$. Then (1) becomes

$$uv - st \leq u - s + v - t.$$

To check that this holds, we notice that it's linear in s, so we only have to check that it holds at $s = 0$ and $s = u$. But this is trivial. On the other hand if $s \leq u$, $v \leq t$, we proceed similarly, getting

$$st + uv - 2sv \leq u - s + t - v,$$

which we again check at $s = 0$ and $s = u$. The rest of the cases are similar to the other two. □

Using the lemma, we get that

$$E(|X|^p) \leq C((s-u)^2 + (t-v)^2)^{p/4} = C((s-u)^2 + (t-v)^2)^{2+(p/4-2)}$$

and Kolmogorov's continuity criterium now implies that the Brownian sheet is α-Hölder continuous for

$$\alpha < \frac{p/4-2}{p} \to \frac{1}{4} \quad (p \to \infty).$$