Stochastic analysis, 9. exercises

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Exercise 1 Let $(B_t : t \geq 0)$ be a Brownian motion in the filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$, which means that B_t and $(B_t^2 - t)$ are continuous $\mathbb{F}$-martingales.

Let $\tau(\omega)$ be a $\mathbb{F}$-stopping time with $E_P(\tau) < \infty$.

(a) Let also $\tau_0 = 0$ and $\tau_n(\omega) = \tau(\omega) \wedge \inf\{t : |B_t(\omega)| > n\}$, $n \in \mathbb{N}$. Show that these are $\mathbb{F}$-stopping times and $\tau_n(\omega) \uparrow \tau(\omega)$ as $n \uparrow \infty$.

(b) Show that for fixed $n \in \mathbb{N}$, $(B^2_{t\wedge\tau_n} - t \wedge \tau_n : t \geq 0)$ is a uniformly integrable $\mathbb{F}$-martingale.

(c) Show that $(B_{\tau_n} : n \in \mathbb{N})$ and $(B^2_{\tau_n} - \tau_n : n \in \mathbb{N})$ are martingales in the discrete time filtration $(\mathcal{F}_{\tau_n} : n \in \mathbb{N})$.

(d) Show that the telescopic sum $B_\tau = \sum_{n=1}^{\infty}(B_{\tau_n} - B_{\tau_{n-1}})$ is convergent in $L^2(P)$.

(e) Show that $E_P(B_\tau^2) = E_P(\tau)$.

Solution 1 (a) Notice that $\{\inf\{t : |B_t(\omega)| > n\} \leq s\} = \bigcup_{t\leq s, t\in\mathbb{Q}}\{|B_t(\omega)| > n\} \in \mathcal{F}_s$, so $\inf\{t : |B_t(\omega)| > n\}$ is a stopping time. The minimum of two stopping times is a stopping time, so $\tau(\omega) \wedge \inf\{t : |B_t(\omega)| > n\}$ is a stopping time. Finally, as $n \uparrow \infty$, $\tau_n(\omega)$ clearly increases and has limit $\tau(\omega)$.

(b) Clearly $|B^2_{t\wedge\tau_n} - t \wedge \tau_n| \leq n^2 + \tau$, which is integrable. We get that the family is U.I. We have shown before that the stopped process is a martingale.

(c) For a fixed $n \in \mathbb{N}$, consider the uniformly integrable martingale $L_t = B_{t\wedge\tau_n}$. We have

$$E(B_{\tau_n}|\mathcal{F}_{\tau_{n-1}}) = E(L_{\tau_n}|\mathcal{F}_{\tau_{n-1}}) = L_{\tau_{n-1}} = M_{\tau_{n-1}}.$$

For the second case, let $L_t = B^2_{t\wedge\tau_n} - t \wedge \tau_n$. Then again

$$E(B^2_{\tau_n} - \tau_n|\mathcal{F}_{\tau_{n-1}}) = E(L_{\tau_n}|\mathcal{F}_{\tau_{n-1}}) = L_{\tau_{n-1}} = B^2_{\tau_{n-1}} - \tau_{n-1}.$$

(d) Notice that since $B^2_{\tau_n} - \tau_n$ is a martingale, $E(B^2_{\tau_n}) = E(\tau_n)$. Moreover,

$$E(B_{\tau_{n-1}}(B_{\tau_n} - B_{\tau_{n-1}})) = E(E(B_{\tau_{n-1}}(B_{\tau_n} - B_{\tau_{n-1}})|\mathcal{F}_{\tau_{n-1}})) = E(B_{\tau_{n-1}}E(B_{\tau_n} - B_{\tau_{n-1}}|\mathcal{F}_{\tau_{n-1}})) = 0.$$

Therefore we have

$$\sum_{n=1}^{\infty} E((B_{\tau_n} - B_{\tau_{n-1}})^2) = \sum_{n=1}^{\infty} \left(E(B_{\tau_n}^2 - B_{\tau_{n-1}}^2 - 2B_{\tau_{n-1}}(B_{\tau_n} - B_{\tau_{n-1}})) \right)$$
$$= \sum_{n=1}^{\infty} (E(\tau_n) - E(\tau_{n-1})) = E(\tau) < \infty.$$

(e) Because the martingale differences are orthogonal,

$$E(B_\tau^2) = \sum_{n=1}^{\infty} E((B_{\tau_n} - B_{\tau_{n-1}})^2) = E(\tau).$$

Exercise 2 We do the same exercise for a continuous local martingale.

Let $(M_t : t \geq 0)$ be a continuous local $\mathbb{F}$-martingale with $M_0 = 0$ and predictable variation $\langle M \rangle_t$, which means that $\langle M \rangle_t$ is $\mathbb{F}$-adapted continuous and non-decreasing with

$$M_t^2 = N_t - \langle M \rangle_t$$

where N_t is a local martingale with $N_0 = \langle M \rangle_0 = 0$.

Let $\sigma_n(\omega) \uparrow \infty$ be a localizing sequence of stopping times such that for each n, the stopped processes $(M_{t\wedge\sigma_n} : t \geq 0)$ and $(N_{t\wedge\sigma_n} : t \geq 0)$ are true $\mathbb{F}$-martingales.

Let $\tau(\omega)$ be an $\mathbb{F}$-stopping time, with $E_P(\langle M \rangle_\tau) < \infty$.

For $n \in \mathbb{N}$ let $\tau_0 = 0$ and

$$\tau_n(\omega) = \tau(\omega) \wedge \sigma_n(\omega) \wedge \inf\{t : |M_t(\omega)| > n\}, \quad n \in \mathbb{N}.$$

(a) Check that $\tau_n(\omega) \uparrow \tau(\omega)$.

(b) Show that for fixed $n \in \mathbb{N}$, $(M_{t\wedge\tau_n}^2 - \langle M \rangle_{t\wedge\tau_n} : t \geq 0)$ is a uniformly integrable $\mathbb{F}$-martingale.

(c) Show that $(M_{\tau_n} : n \in \mathbb{N}$ and $(M_{\tau_n}^2 - \langle M \rangle_{\tau_n} : n \in \mathbb{N})$ are martingales in the discrete time filtration $(\mathcal{F}_{\tau_n} : n \in \mathbb{N})$.

(d) Show that the telescopic sum $M_\tau = \sum_{n=1}^{\infty}(M_{\tau_n} - M_{\tau_{n-1}})$ is convergent in $L^2(P)$.

Solution 2 (a) Since $\sigma_n(\omega) \uparrow \infty$ and $\inf\{t : |M_t(\omega)| > n\} \uparrow \infty$, $\tau_n(\omega) \uparrow \tau(\omega)$.

(b) We have $|M_{t\wedge\tau_n}^2 - \langle M \rangle_{t\wedge\tau_n}| \leq n^2 + M_\tau$, which is integrable. Therefore $M_{t\wedge\tau_n}^2 - \langle M \rangle_{t\wedge\tau_n} = N_{t\wedge\tau_n}$ is a U.I. martingale.

(c) For a fixed $n \in \mathbb{N}$, consider the uniformly integrable martingale $L_t = M_{t\wedge\tau_n}$. We have

$$E(M_{\tau_n} | \mathcal{F}_{\tau_{n-1}}) = E(L_{\tau_n} | \mathcal{F}_{\tau_{n-1}}) = L_{\tau_{n-1}} = M_{\tau_{n-1}}.$$

For the second case, let $L_t = M_{t\wedge\tau_n}^2 - \langle M \rangle_{t\wedge\tau_n}$. Then again

$$E(M_{\tau_n}^2 - \langle M \rangle_{\tau_n} | \mathcal{F}_{\tau_{n-1}}) = E(L_{\tau_n} | \mathcal{F}_{\tau_{n-1}}) = L_{\tau_{n-1}} = M_{\tau_{n-1}}^2 - \langle M \rangle_{\tau_{n-1}}.$$

(d) We have

$$\begin{aligned}\sum_{n=1}^{\infty} E((M_{\tau_n}-M_{\tau_{n-1}})^2) &= \sum_{n=1}^{\infty}\left(E(M^2_{\tau_n})-E(M^2_{\tau_{n-1}})-2E(M_{\tau_{n-1}}(M_{\tau_n}-M_{\tau_{n-1}}))\right)\\ &= \sum_{n=1}^{\infty}\left(E(\langle M\rangle_{\tau_n})-E(\langle M\rangle_{\tau_{n-1}})-2E(E(M_{\tau_{n-1}}(M_{\tau_n}-M_{\tau_{n-1}})|\mathcal{F}_{\tau_{n-1}}))\right)\\ &= \sum_{n=1}^{\infty}\left(E(\langle M\rangle_{\tau_n})-E(\langle M\rangle_{\tau_{n-1}})\right) = E(\langle M\rangle_{\tau}) < \infty.\end{aligned}$$

(e) The martingale differences are orthogonal, so by (d) we have $E(M^2_\tau) = E(\langle M\rangle_\tau)$.

Exercise 3 If $(M'_t : t \geq 0)$ is another continuous local martingale, with $M_0' = 0$, and τ is a stopping time with $E(\tau) < \infty$, show that

$$E(M_\tau M'_\tau) = E(\langle M, M'\rangle_\tau),$$

where the predictable covariation $\langle M, N\rangle_t$ is the continuous $\mathbb{F}$-adapted process of finite variation on compacts such that

$$M_t M'_t - M_0 M'_0 - \langle M, M'\rangle_t$$

is a local $\mathbb{F}$-martingale.

Solution 3 Notice that the previous exercise applied to the local martingales $M_t + M'_t$ and $M_t - M'_t$ yields

$$E(M_\tau M'_\tau) = E\left(\frac{1}{4}\left((M_\tau + M'_\tau)^2 - (M_\tau - M'_\tau)^2\right)\right) = E\left(\frac{1}{4}\left(\langle M+M'\rangle_\tau - \langle M-M'\rangle_\tau\right)\right) = E(\langle M, M'\rangle).$$

Exercise 4 Let $M_t(\omega)$ be adapted to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, with $E(|M_t|) < \infty$ for all t. Then (M_t) is an $\mathbb{F}$-martingale if and only if for all $\mathbb{F}$-stopping times $\tau(\omega)$ which can take at most two finite values

$$E(M_\tau) = E(M_0).$$

Solution 4 Let $s \leq t$ and $A \in \mathcal{F}_s$. Define $\tau = t\mathbf{1}_A + s\mathbf{1}_{A^c}$. Then τ is a stopping time, since

$$\{\tau(\omega) \leq u\} = \{t\mathbf{1}_A + s\mathbf{1}_{A^c} \leq u\} = \begin{cases} \emptyset, & \text{if } u < s \\ A^c, & \text{if } s \leq u < t \\ \Omega, & \text{if } u \geq t \end{cases}$$

from which we see that $\{\tau(\omega) \leq u\} \in \mathcal{F}_u$. Therefore

$$E((M_t - M_s)\mathbf{1}_A) = E(M_\tau \mathbf{1}_A - M_s \mathbf{1}_A) = E(M_\tau - M_\tau \mathbf{1}_{A^c} - M_s \mathbf{1}_A) = E(M_\tau - M_s) = E(M_0) - E(M_0) = 0.$$