Stochastic analysis, 13. exercises

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Exercise 1 (Bougerol's identity) Consider $\sinh(W_t)$, where W_t is Brownian motion.

Let also B_t and β_t be independent Brownian motions, and

$$X_t = e^{B_t} \int_0^t e^{-B_s}\, d\beta_s.$$

Apply Ito formula to show that the process $(X_t : t \geq 0)$ and $(\sinh(W_t) : t \geq 0)$ have the same distributions, i.e. they satisfy the same stochastic differential equation in Ito sense.

Solution 1 Let $f(x) = \sinh(x)$. Then $f'(x) = \cosh(x)$ and $f''(x) = \sinh(x)$. Thus by Ito formula,

$$\sinh(W_t) = \int_0^t \cosh(W_s)\, dW_s + \frac{1}{2} \int_0^t \sinh(W_s)\, ds.$$

It follows that $Y_t = \sinh(W_t)$ satisfies the SDE

$$dY_t = \sqrt{1 + Y_t^2}\, dW_t + \frac{1}{2} Y_t dt$$

On the other hand, integration by parts gives

$$\begin{aligned} dX_t &= \left(\int_0^t e^{-B_s}\, d\beta_s \right) d(e^{B_t}) + e^{B_t} d \left(\int_0^t e^{-B_s}\, d\beta_s \right) + \langle e^{B_t}, \int_0^t e^{-B_s}\, d\beta_s \rangle_t \\ &= e^{B_t} \left(\int_0^t e^{-B_s}\, d\beta_s \right) dB_t + \frac{1}{2} e^{B_t} \left(\int_0^t e^{-B_s}\, d\beta_s \right) dt + e^{B_t} e^{-B_t} d\beta_t \\ &= X_t dB_t + d\beta_t + \frac{1}{2} X_t dt. \end{aligned}$$

Here we have used the fact that $d(e^{B_t}) = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt$ and $\langle e^{B_t}, \int_0^t e^{-B_s}\, d\beta_s \rangle_t = \int_0^t e^{-B_s} d\langle \beta_s, e^{B_s} \rangle_s = 0$. The latter fact follows because $\langle \beta_t, e^{B_t} \rangle_t = \int_0^t e^{B_s}\, d\langle B_s, \beta_s \rangle_s = 0$. Finally we notice that if we define a process Z_t by the SDE $X_t dB_t + d\beta_t = \sqrt{1 + X_t^2} dZ_t$, then Z_t is a Brownian motion since

$$
\begin{aligned}
\langle Z\rangle_t &= \langle \int_0^t \frac{X_s}{\sqrt{1+X_s^2}}\, dB_s + \int_0^t \frac{1}{\sqrt{1+X_s^2}}\, d\beta_s\rangle_t \\
&= \int_0^t \frac{X_s^2}{1+X_s^2}\, dt + \int_0^t \frac{1}{1+X_s^2}\, dt + \langle \int_0^t \frac{X_s}{\sqrt{1+X_s^2}}\, dB_s, \int_0^t \frac{1}{\sqrt{1+X_s^2}}\, d\beta_s\rangle \\
&= t + \int_0^t \frac{1}{\sqrt{1+X_s^2}}\, d\langle \int_0^t \frac{X_s}{\sqrt{1+X_s^2}}\, dB_s, \beta_s\rangle_s \\
&= t.
\end{aligned}
$$

Exercise 2 We consider a $\mathbb{C}$-valued continuous local martingale

$$Z_t = X_t + iY_t$$

where X_t and Y_t are $\mathbb{R}$-valued continuous local martingales in a filtration $\mathbb{F}$.

(a) Prove that there is an unique continuous $\mathbb{C}$-valued process of finite variation $\langle Z, Z\rangle_t$ such that

$$Z_t^2 - \langle Z, Z\rangle_t$$

is a complex local martingale.

(b) Prove that the following statements are equivalent:

i. Z_t^2 is a $\mathbb{F}$-local martingale

ii. $\langle Z, Z\rangle_t = 0$ for all t

iii. $\langle X, X\rangle_t = \langle Y, Y\rangle$ and $\langle X, Y\rangle = 0$

Such $\mathbb{C}$-valued martingales are called *conformal* local martingales. For example, if B_t and β_t are independent $\mathbb{R}$-valued Brownian motions in the filtration $\mathbb{F}$, the $\mathbb{C}$-valued *planar Brownian motion*

$$W_t = B_t + i\beta_t$$

is a conformal martingale.

(c) Use Lévy characterization theorem to show that if $Z_t = X_t + iY_t$ is a $\mathbb{C}$-valued continuous conformal local martingale in a filtration $\mathbb{F}$, there exists a $\mathbb{C}$-valued planar Brownian motion, such that

$$Z_t = W_{\langle X, X\rangle_t}$$

with

$$W_u = X_{\tau(u)} + iY_{\tau(u)}, \quad \tau(u) = \inf\{t : \langle X, X\rangle_t \geq u\}.$$

(d) Show that for a conformal local martingale $Z_t = X_t + iY_t$

$$\langle \Re(Z), \Re(Z)\rangle_t = \frac{1}{2}\langle Z, \overline{Z}\rangle_t$$

(e) Let H_t be a $\mathbb{C}$-valued bounded progressive process, and Z_t a conformal local martingale. Then

$$U_t = \int_0^t H_s\, dZ_s = \int_0^t \Re(H_s)\, dX_s - \int_0^t \Im(H_s)\, dY_s + i\left(\int_0^t \Im(H_s)\, dX_s + \int_0^t \Re(H_s)\, dY_s \right)$$

is a conformal martingale with

$$\langle U, \overline{U} \rangle_t = \int_0^t |H_s|\, d\langle Z, \overline{Z} \rangle_t$$

(f) Let Z_t be a continuous conformal local martingale and $F\colon \mathbb{C} \to \mathbb{C}$ twice differentiable as a function of two real variables. Use Ito formula to show

$$F(Z_t) = F(Z_0) + \int_0^t \frac{\partial}{\partial z} F(Z_s)\, dZ_s + \int_0^t \frac{\partial}{\partial \overline{z}} F(Z_s) d\overline{Z}_s + \frac{1}{4} \int_0^t \Delta F(Z_s)\, d\langle Z, \overline{Z}\, \rangle_s$$

where for $z = x + iy$

$$\begin{aligned} \Delta F(z) = \Delta(F(x,y)) &= \frac{\partial^2}{\partial x^2} F(x,y) + \frac{\partial^2}{\partial y^2} F(x,y) \\ &= \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) F(x,y) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} F(x,y). \end{aligned}$$

(g) We say that F is harmonic if $\Delta F(z) = 0$ for all $z \in \mathbb{C}$. Show that if F is harmonic and Z_t is a continuous conformal local martingale, then $F(Z_t)$ is a local martingale.

(h) Let $F(z)$ be holomorphic. Show that if Z_t is a continuous conformal local martingale, then $F(Z_t)$ is a continuous conformal local martingale with

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_s)\, dZ_s.$$

(i) Show that if W_t is a $\mathbb{C}$-valued planar Brownian motion and $F\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and non-constant, then

$$F(W_t) = F(W_0) + \widetilde{W}_{\langle X, X \rangle_t}$$

where $X_t = \Re(F(W_t))$ and

$$\langle X, X \rangle_t = \int_0^t |F'(W_s)|^2\, ds$$

and $\widetilde{W}_t$ is a $\mathbb{C}$-valued planar Brownian motion.

In other words $F(W_t)$ is a time changed Brownian motion. We say that Brownian motion is *conformal invariant*.

Solution 2 (a) Since X_t and Y_t are local martingales, there are unique continuous processes $\langle X \rangle_t$, $\langle Y \rangle_t$ and $\langle X, Y \rangle_t$ such that the processes

$$X_t^2 - \langle X \rangle_t, \quad Y_t^2 - \langle Y \rangle_t, \quad X_t Y_t - \langle X, Y \rangle_t$$

are continuous local martingales. Now

$$Z_t^2 - \langle X \rangle_t + \langle Y \rangle_t - 2i\langle X, Y \rangle_t = (X_t^2 - \langle X \rangle_t) - (Y_t^2 - \langle Y \rangle_t) + 2i(X_t Y_t - \langle X, Y \rangle_t)$$

is a complex local martingale, so we may define

$$\langle Z, Z \rangle_t = \langle X \rangle_t - \langle Y \rangle_t + 2i\langle X, Y \rangle_t. \tag{1}$$

The uniqueness of this process of finite variation follows from the real case by noticing that the real and imaginary parts must be unique separately.

(b) Assume first that Z_t^2 is a local martingale. Then by uniqueness, $\langle Z, Z \rangle_t = 0$ for all t. On the other hand if $\langle Z, Z \rangle_t = 0$ for all t, then by (1) we have $\langle X, X \rangle_t = \langle Y, Y \rangle_t$ and $\langle X, Y \rangle_t = 0$. Finally if (iii) holds, then clearly $\langle Z \rangle_t = 0$, so Z_t^2 is a local martingale.

(c) By Dambis, Dubins-Schwartz, the process $X_{\tau(u)}$ is a Brownian motion, and since Z_t is conformal, also $Y_{\tau(u)}$ is a Brownian motion. Hence W_u is a planar Brownian motion.

(d) We have

$$\langle Z, \overline{Z} \rangle_t = \langle X + iY, X - iY \rangle_t = \langle X, X \rangle_t + \langle Y, Y \rangle_t = 2\langle \Re(Z), \Re(Z) \rangle_t.$$

(e) U_t is clearly a martingale, since its real and imaginary parts are martingales. Moreover, since Z_t is a conformal martingale,

$$\begin{aligned}
\langle U, U \rangle_t = {} & \langle \int_0^t \Re(H_s)\, dX_s \rangle_t + \langle \int_0^t \Im(H_s)\, dY_s \rangle_t - \langle \int_0^t \Im(H_s)\, dX_s \rangle_t - \langle \int_0^t \Re(H_s)\, dY_s \rangle \\
& + 2i \left(\langle \int_0^t \Re(H_s)\, dX_s, \int_0^t \Im(H_s)\, dX_s \rangle_t - \langle \int_0^t \Im(H_s)\, dY_s, \int_0^t \Re(H_s)\, dY_s \rangle_t \right) \\
= {} & 0.
\end{aligned}$$

Thus U is a conformal local martingale. Finally,

$$\begin{aligned}
\langle U, \overline{U} \rangle_t &= 2\langle \int_0^t \Re(H_s)\, dX_s - \int_0^t \Im(H_s)\, dY_s \rangle_t \\
&= 2\langle \int_0^t \Re(H_s)\, dX_s \rangle_t + 2\langle \int_0^t \Im(H_s)\, dY_s \rangle_t \\
&= 2 \int_0^t \Re(H_s)^2\, d\langle X \rangle_s + 2 \int_0^t \Im(H_s)^2\, d\langle Y \rangle_s \\
&= \int_0^t |H_s|^2\, d\langle Z, \overline{Z} \rangle_s
\end{aligned}$$

(f) By Ito formula we have

$$
\begin{aligned}
F(Z_t) &= F(Z_0) + \int_0^t \frac{\partial}{\partial x} F(Z_s)\, dX_s + \int_0^t \frac{\partial}{\partial y} F(Z_s)\, dY_s + \int_0^t \frac{\partial^2}{\partial x \partial y} F(Z_s)\, d\langle X, Y\rangle_s \\
&\quad + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} F(Z_s)\, d\langle X\rangle_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2} F(Z_s)\, d\langle Y\rangle_s \\
&= F(Z_0) + \int_0^t \frac{\partial}{\partial x} F(Z_s)\, d\left(\frac{Z_s + \overline{Z}_s}{2}\right) + \int_0^t \frac{\partial}{\partial y} F(Z_s)\, d\left(\frac{Z_s - \overline{Z}_s}{2i}\right) + \frac{1}{2}\int_0^t \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F(Z_s)\, d\langle X\rangle_s \\
&= F(Z_0) + \int_0^t \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) F(Z_s)\, dZ_s + \int_0^t \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) F(Z_s)\, d\overline{Z}_s + \frac{1}{4}\int_0^t \Delta F(Z_s)\, d\langle Z, \overline{Z}\rangle_s \\
&= F(Z_0) + \int_0^t \frac{\partial}{\partial z} F(Z_s)\, dZ_s + \int_0^t \frac{\partial}{\partial \overline{z}} F(Z_s)\, d\overline{Z}_s + \frac{1}{4}\int_0^t \Delta F(Z_s)\, d\langle Z, \overline{Z}\rangle_s.
\end{aligned}
$$

(g) This follows directly from the Ito formula above since the non-martingale part vanishes.

(h) By (g), $F(Z_t)$ is a continuous conformal local martingale. By using (f), we have

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_s)\, dZ_s.$$

(i) We have

$$F(W_t) = F(W_0) + \int_0^t F'(W_s)\, dW_s.$$

Let $Y_t = \int_0^t F'(W_s)\, dW_s$. By (c) there exists a $\mathbb{C}$-valued planar Brownian motion $\widetilde{W}_u$ such that $\widetilde{W}_{\langle \Re(Y), \Re(Y)\rangle_t} = Y_t$. Now by (e) we have

$$\langle \Re(Y), \Re(Y)\rangle_t = \frac{1}{2}\langle Y, \overline{Y}\rangle_t = \frac{1}{2}\int_0^t |F'(W_s)|^2\, dW_s = \langle X, X\rangle_t$$

and hence

$$F(W_t) = F(W_0) + \widetilde{W}_{\langle X, X\rangle_t}.$$