Stochastic analysis, spring 2013, Exercises-9, 28.03.2013

1. Let $(B_t : t \geq 0)$ a Brownian motion in the filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ which means that B_t and $(B_t^2 - t)$ are continuous $\mathbb{F}$-martingales.

   Let $\tau(\omega)$ be a $\mathbb{F}$-stopping time with $E_P(\tau) < \infty$.

   (a) Let also $\tau_0 = 0$ and $\tau_n(\omega) = \tau(\omega) \wedge \inf\{t : |B_t(\omega)| > n\}, \quad n \in \mathbb{N}$. Show that these are $\mathbb{F}$-stopping times and $\tau_n(\omega) \uparrow \tau(\omega)$ as $n \uparrow \infty$.

   (b) Show that for fixed $n \in \mathbb{N}$, $\big(B^2_{t\wedge\tau_n} - t \wedge \tau_n\big) : t \geq 0\big)$ is an uniformly integrable $\mathbb{F}$-martingale.

   (c) Show that $\big(B_{\tau_n} : n \in \mathbb{N}\big)$ and $\big((B^2_{\tau_n} - \tau_n : n \in \mathbb{N}\big)$ are martingales in the discrete time filtration $(\mathcal{F}_{\tau_n} : n \in \mathbb{N})$.

   (d) Show that the telescopic sum $B_\tau = \sum_{n=1}^\infty (B_{\tau_n} - B_{\tau_{n-1}})$ is convergent in $L^2(P)$.
   Hint: by the discrete integration by parts formula

   $$\big(B_{\tau_n} - B_{\tau_{n-1}}\big)^2 = B^2_{\tau_n} - B^2_{\tau_{n-1}} - 2B_{\tau_{n-1}}(B_{\tau_n} - B_{\tau_{n-1}}\big)$$

   then use the martingale properties.

   (e) Show that $E_P(B^2_\tau) = E_P(\tau)$.

2. We do the same exercise for a continuous local martingale.

   Let $(M_t : t \geq 0)$ a continuous local $\mathbb{F}$-martingale with $M_0 = 0$ and predictable variation $\langle M\rangle_t$, which means that $\langle M\rangle_t$ is $\mathbb{F}$-adapted continuous and non-decreasing with

   $$M_t^2 = N_t - \langle M\rangle_t$$

   where N_t is a local martingale with $N_0 = \langle M\rangle_0 = 0$.

   Let $\sigma_n(\omega) \uparrow \infty$ a localizing sequence of stopping times such that for each n, the stopped processes $(M_{t\wedge\sigma_n} : t \geq 0)$ and $(N_{t\wedge\sigma_n} : t \geq 0)$ are true $\mathbb{F}$-martingales.

   Let $\tau(\omega)$ an $\mathbb{F}$-stopping time, with $E_P\big(\langle M\rangle_\tau\big) < \infty$.

   For $n \in \mathbb{N}$ let $\tau_0 = 0$ and

   $$\tau_n(\omega) = \tau(\omega) \wedge \sigma_n(\omega) \wedge \inf\{t : |M_t(\omega)| > n\}, \quad n \in \mathbb{N}.$$

   (a) Check that $\tau_n(\omega) \uparrow \tau(\omega)$.

   (b) Show that for fixed $n \in \mathbb{N}$, $\big(M^2_{t\wedge\tau_n} - \langle M\rangle_{t\wedge\tau_n}\big) : t \geq 0\big)$ is an uniformly integrable $\mathbb{F}$-martingale.

   (c) Show that $\big(M_{\tau_n} : n \in \mathbb{N}\big)$ and $\big((M^2_{\tau_n} - \langle M\rangle_{\tau_n} : n \in \mathbb{N}\big)$ are martingales in the discrete time filtration $(\mathcal{F}_{\tau_n} : n \in \mathbb{N})$.

   (d) Show that the telescopic sum $M_\tau = \sum_{n=1}^\infty (M_{\tau_n} - M_{\tau_{n-1}})$ is convergent in $L^2(P)$.

   $$\big(M_{\tau_n} - M_{\tau_{n-1}}\big)^2 = M^2_{\tau_n} - M^2_{\tau_{n-1}} - 2M_{\tau_{n-1}}(M_{\tau_n} - M_{\tau_{n-1}}\big)$$

   Hint: by the discrete integration by parts formula and the martingale properties.

(e) Show that $E_P(M_\tau^2) = E_P(\langle M\rangle_\tau)$.

3. If $(M'_t : t \ge 0)$ is another continuous local martingale, with $M'_0 = 0$, and τ is a stopping time with $E(\tau) < \infty$, show that

$$E_P\big(M_\tau M'_\tau\big) = E_P\big(\langle M, N\rangle_\tau\big) \tag{1}$$

where the predictable covariation $\langle M, N\rangle_t$ is the continuous $\mathbb{F}$-adapted process of finite variation on compacts such that

$$M_t M'_t - M_0 M'_0 - \langle M, M'\rangle_t$$

is a $\mathbb{F}$-local martingale.

Hint: you can use polarization : $MM' = \big((M+M')^2 - (M-M')^2\big)/4$.

4. (Martingale characterization) Let $M_t(\omega)$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, with $E(|M_t|) < \infty\ \forall t$. Then (X_t) is an $\mathbb{F}$-martingale if and only if $\forall$ $\mathbb{F}$-stopping times $\tau(\omega)$ which can take at most two finite values

$$E(M_\tau) = E(M_0)$$

Hint: the necessity is just Doob optional sampling theorem, for sufficency for $s \le t$ and $A \in \mathcal{F}_s$ define $\tau(\omega) = \big(t\mathbf{1}_A + s\mathbf{1}_{A^c}\big)$ and show that it is a stopping time. Then show that $E\big((M_t - M_s)\mathbf{1}_A\big) = 0$.