Stochastic analysis, spring 2013, Exercises-7, 14.03.2013

1. Suppose we have an urn which contains at time $t = 0$ two balls, one black and one white. At each time $t \in N$ we draw uniformly at random from the urn one ball, and we put it back together with a new ball of the same colour.

   We introduce the random variables
   $X_t(\omega) = \mathbf{1}\{$ the ball drawn at time t is black $\}$
   and denote $S_t = (1 + X_1 + \cdots + X_t)$,
   $M_t = S_t/(t + 2)$, the proportion of black balls in the urn.
   We use the filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_n = \sigma\{X_s : s \in \mathbb{N}, s \le t\}$.

   i) Compute the Doob decomposition of (S_t), $S_t = S_0 + N_t + A_t$, where (N_t) is a martingale and (A_t) is predictable.

   ii) Show that (M_t) is a martingale and find the representation of (M_t) as a martingale transform $M_t = (C \cdot N)_t$, where (N_t) is the martingale part of (S_t) and (C_t) is predictable.

   iv) Note that the martingale $(M_t)_{t \ge 0}$ is uniformly integrable (Why ?). Show that P a.s. and in L^1 exists $M_\infty = \lim_{t\to\infty} M_t$. Compute $E(M_\infty)$.

   v) Show that $P(0 < M_\infty < 1) > 0$.

   Since $M_\infty(\omega) \in [0, 1]$, it is enough to show that $0 < E(M_\infty^2) < E(M_\infty)$ with strict inequalities.

   Hint: compute the Doob decomposition of the submartingale (M_t^2), and than take expectations before going to the limit to find the value of $E(M_\infty^2)$.

2. Consider an i.i.d. random sequence $(U_t : t \in \mathbb{N})$ with uniform distribution on $[0, 1]$, $P(U_1 \in dx) = \mathbf{1}_{[0,1]}(x)dx$. Note that $E_P(U_t) = 1/2$.

   Consider also the random variable $-\log(U_1(\omega))$ which is 1-exponential w.r.t. P.

   $$P(-\log(U_1) > x) = \begin{cases} \exp(-x) & \text{kun } x \ge 0 \\ 1 & \text{kun } x < 0 \end{cases}$$

   $-\log(U_1) \in L^1(P)$ with $E_P(-\log(U_1)) = 1$.

   (a) Let $Z_0 = 1$, and

   $$Z_t(\omega) = 2^t \prod_{s=1}^{t} U_s(\omega)$$

   Show that (Z_t) is a martingale in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, with $\mathcal{F}_t = \sigma(Z_1, Z_2, \ldots, Z_t) = \sigma(U_1, U_2, \ldots, U_t)$.

(b) Show that $E_P(Z_t) = 1$.

(c) Show that the limit $Z_\infty(\omega) = \lim_{t\to\infty} Z_t(\omega)$ exists P almost surely.

(d) Show that

$$Z_\infty(\omega) = 0 \qquad P\text{-a.s.}$$

Hint Compute first the P-a.s. limit

$$\lim_{t\to\infty} \frac{1}{t} \log(Z_t(\omega))$$

(remember Kolmogorov's strong law of large numbers!).

(e) Show that the martingale $(Z_t(\omega) : t \in \mathbb{N})$ is not uniformly integrable.

(f) Show that $\log(Z_t(\omega))$ is a supermartingale, does it satisfy the assumptions of Doob's martingale convergence theorem ?

(g) At every time $t \in \mathbb{N}$, define the probability measure

$$Q_t(A) := E_P(Z_t \mathbf{1}_A) \qquad \forall A \in \mathcal{F}_t$$

on the probability space $(\Omega, \mathcal{F})$.

Show that the random variables $(U_1, \dots, U_t)$ are i.i.d. also under Q_t, compute their probability density under Q_t.

Hint: use Doob's maximal inequality.

3. (Paley's and Littlewood's maximal function) Consider a function in $f(x) \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$.

Define the σ-algebra

$$\mathcal{F}_k = \sigma\big\{ Q_{k,z} = \big(z2^{-k}, (z+1)2^{-k}\big],\ z \in \mathbb{Z}^d \big\} \subseteq \mathcal{B}(\mathbb{R}^d), \quad k \in \mathbb{Z}$$

and the two sided filtration $\mathbb{F} = (\mathcal{F}_k : k \in \mathbb{Z})$ where the dyadic cubes $(Q_{k,z} : z \in \mathbb{Z}^d)$ form a partition of $\mathbb{R}^d$, and the functions

$$f_k(x) = \sum_{z\in\mathbb{Z}^d} \mathbf{1}(x \in Q_{k,z}) \frac{1}{|Q_{k,z}|} \int_{Q_{k,z}} f(y)dy$$

where for $k \in \mathbb{Z}$, $|Q_{k,z}| = 2^{-kd}$ is the Lebesgue measure of the d-dimensional dyadic cube

Show that $f_k(x)$ is an $\mathbb{F}$-martingale w.r.t. Lebesgue measure. Note that the definition of conditional expectation martingales extends directly to the case where we integrate with respect to σ-finite positive measures. To work with a probability measure, we could take instead with $f(x) \in L^1([0,1]^d, \mathcal{B}([0,1]^d), dx)$.

Show that $\lim_{k\to-\infty} f_k(x) = 0$ both almost surely w.r.t. the Lebesgue measure but not $L^1(\mathbb{R}^d)$-sense.

Define the *maximal function*

$$f^{\square}(x) := \sup_{k\in\mathbb{Z}} f_k(x)$$

Show that for $1 < p < \infty$

$$\| f^{\square}(x) \|_{L^p(\mathbb{R}^d)} \leq \frac{p}{p-1} \sup_{k \in \mathbb{Z}} \| f_k \|_{L^p(\mathbb{R}^d)} \leq \frac{p}{p-1} \| f \|_{L^p(\mathbb{R}^d)}$$

and

$$cP(|f^{\square}(x)| > c) \leq \sup_{k \in \mathbb{Z}} \| f_k \|_{L^1(\mathbb{R}^d)} \leq \| f \|_{L^1(\mathbb{R}^d)}$$