Stochastic analysis, spring 2013, Exercises-6, 28.02.2013

A branching process $(Z_t)_{t\in\mathbb{N}}$ with integer values, represents the size of a population evolving randomly in discrete time.

We start with $Z_0(\omega) = 1$ individual at time $t = 0$.

Inductively each of the $Z_{t-1}(\omega)$ individuals in the $(t-1)$ generation has a random number of offspring $Y_{i,t}$. These offspring numbers are independent and identically distributed with law $\pi = (\pi(n) : n = 0, 1, \dots)$,

$\pi(n) = P\big(Y = n\big)$, $Y = Y_{1,1}$.

The size of the new generation at time t is then

$$Z_t(\omega) = \sum_{i=1}^{Z_{t-1}(\omega)} Y_{i,t}(\omega)$$

We assume that the mean offspring number is finite

$$\mu = E_\pi(Y) = \sum_{n=0}^{\infty} n\pi(n) < \infty$$

Note that if $Z_t(\omega) = 0$, then $Z_u(\omega) = 0 \;\forall u \geq t$. In this case we say that the process is extinct. Clearly $P(Z_t = 0) \leq P(Z_u = 0)$ for $t \leq u$.

Note also that $P(Y = 0) > 0$ implies $P(Z_t = 0) > 0$, $\forall t \geq 1$.

Consider the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$ with $\mathcal{F}_t = \sigma(Z_s : 0 \leq s \leq t)$.

Actually we could consider the larger filtration $\mathbb{F}' = (\mathcal{F}'_t : t \in \mathbb{N})$ with

$$\mathcal{F}'_t = \sigma\big(Z_0, Y_{s,i}\mathbf{1}(Z_{s-1} \geq i) : 0 \leq s \leq t,\; i \in \mathbb{N}\big).$$

or $\mathbb{F}'' = (\mathcal{F}''_t : t \in \mathbb{N})$ with

$$\mathcal{F}''_t = \sigma\big(Z_0, Y_{s,i} : 0 \leq s \leq t,\; i \in \mathbb{N}\big).$$

Although $\mathcal{F}_t \subset \mathcal{F}'_t \subset \mathcal{F}''_t$, the martingale properties we use in this exercise for all these filtrations.

1. Show that $Z_t(\omega)$ is a $\mathbb{F}$-martingale, (respectively supermartingale, submartingale) when $\mu = 1$ (respectively $0 \leq \mu < 1$, $1 < \mu < \infty$, in the filtration generated by the process Z itself.

2. For $\mu \neq 1$, write the Doob decomposition of the supermartingale (respectively martingale) Z_t as sum of a martingale and a non-increasing (respectively non-decreasing) $\mathbb{F}$-predictable process, and compute the mean $E(Z_t)$ for $t \in \mathbb{N}$.

3. Assume that $\mu \leq 1$, and that the offspring distribution is non-trivial, meaning that $0 \leq \pi(Y = 1) < 1$. The case $P(Y = 1) = 1$ is trivial, nothing happens, the size of the population is constant.

   Show that when $\mu \leq 1$ (subcritical and critical cases)

   $$\lim_{t\to\infty} Z_t(\omega) = 0 \qquad P \text{ a.s.}$$

Hint: first show that a finite limit $Z_\infty(\omega)$ exists P a.s. with $E(Z_\infty) < \infty$. Use the indepdence of $Y_{1,1}$ from $(Y_{t,i} : t \geq 2, i \in \mathbb{N})$ to prove

$$P(Z_\infty = 0|Z_1 = n) = P(Z_\infty = 0)^n$$

where $P(Z_\infty = 0)$ is the probability that the descendance of a single individual becomes extinct.

By computing first the conditional probability $P(Z_\infty = 0|\sigma(Z_1))(\omega)$ and taking expectation, show that the unknown $q = P(Z_\infty = 0)$ satisfies the equation

$$q = E_P(q^Y), \quad q \in [\pi(0), 1]$$

where $P(Y = n) = \pi(n)$ is the offspring distribution.

Note that since $\mu = E(Y) \leq 1$ and $\pi(1) = P(Y = 1) < 1$, necessarily $\pi(0) = P(Y = 0) > 0$, and $P(Z_\infty = 0) \geq P(Y = 0) > 0$. Therefore $q = 0$ is not a solution.

$q = 1$ is also a solution. We show that there are no other solutions.

Hint take the the derivative

$$\frac{d}{dq} E_P(q^Y)$$

checking that it is allowed to take a derivative inside the expectation, and show that $q < 1$ is in contradiction with $E_P(q^Y) = q$.

4. In the critical case $\mu = 1$, show that the martingale $(Z_t : t \in \mathbb{N})$ is not uniformly integrable

5. Next we work with the supercritical case, with $\mu = E_P(Y) \in (1, \infty)$.

6. Show that

$$W_t = Z_t(\omega)\mu^{-t}$$

is a martingale.

7. Show that P almost surely $\lim_{t\to\infty} W_t \to W_\infty$ with $W_\infty \in L^1(P)$.

8. The next result is a theorem from Kesten and Stigum (1966) which states that W_t is an uniformly integrable martingale if and only if the offspring distribution satisfies

$$E_P(Y \log(Y)) = 0$$

where it is understood that $0 \log(0) = \lim_{x\downarrow 0} x \log(x) = 0$.

Write the increments:

$$W_t - W_{t-1} = \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-i}} (Y_{t,i} - \mu)$$

and truncate them in the following way: for

$$\widetilde{W}_t = \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-i}} Y_{t,i} \mathbf{1}(Y_{t,i} \le \mu^t),$$

$$R_t = \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-i}} E\Big(Y \mathbf{1}(Y > \mu^t) \Big)$$

We decompose

$$W_t - W_{t-1} = \underbrace{\Big(W_t - \widetilde{W}_t - R_t \Big)}_{\text{I}} + \underbrace{\Big(\widetilde{W}_t + R_t - W_{t-1} \Big)}_{\text{II}} =$$

$$\frac{1}{\mu^t} \sum_{i=1}^{Z_{t-1}} \Big\{ Y_{t,i} \mathbf{1}(Y_{t,i} > \mu^t) - E\big(Y \mathbf{1}(Y > \mu_t) \big) \Big\} + \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-1}} \Big\{ Y_{t,i} \mathbf{1}(Y_{t,i} \le \mu^t) - E\big(Y \mathbf{1}(Y \le \mu_t) \big) \Big\}$$

where (I) and (II) are martingale differences.

9. Show that

$$E\Bigg(\Bigg\{ \sum_{t=1}^{\infty} \frac{1}{\mu^t} \sum_{i=1}^{Z_{t-1}} \Big(Y_{t,i} \mathbf{1}(Y_{t,i} \le \mu^t) - E\big(Y \mathbf{1}(Y \le \mu^t) \big) \Big) \Bigg\}^2 \Bigg) < \infty$$

   Therefore by summing the increments (I), we obtain a martingale bounded in $L^2(P)$ which is also uniformly integrable.

10. Show also that, when $1 < E(Y) < \infty$, without any additional assumptions

$$\sum_{t=1}^{\infty} P(\widetilde{W}_t \ne W_t) < \infty$$

   and by the Borel Cantelli lemma, with probability one $\widetilde{W}_t \ne W_t$ only for finitely many t.

11. Show that the series

$$\sum_{t=1}^{\infty} \mu^{-t} \sum_{i=1}^{Z_{t-1}} \Big\{ Y_{t,i} \mathbf{1}(Y_{t,i} > \mu^t) - E_P\big(Y \mathbf{1}(Y > \mu^t) \big) \Big\}$$

   converges in $L^1(P)$ if and only if $E_P(Y \log Y) < \infty$.

   Hint: it is enough to show that

$$\sum_{t=1}^{\infty} \mu^{-t} E_P\Bigg(\sum_{i=1}^{Z_{t-1}} \Big\{ Y_{t,i} \mathbf{1}(Y_{t,i} > \mu^t) \Bigg) < \infty$$

12. Show that when $1 < E(Y) < \infty$, W_t is uniformly integrable if and only if $E_P(Y \log Y) < \infty$.