Stochastic analysis, spring 2013, Exercises-5, 21.02.2013

1. We have seen that when $E_P(|X|) < \infty$,

$$\forall \varepsilon > 0,\ \exists \delta :\ P(A) < \delta \Longrightarrow E_P\big(|X|\mathbf{1}_A\big) < \varepsilon$$

Show that a collection $\mathcal{C} \subset L^1(P)$ which is **bounded** in $L^1(P)$, is uniformly integrable if and only if

$$\forall \varepsilon > 0,\ \exists \delta :\ P(A) < \delta \Longrightarrow \sup_{X \in \mathcal{C}} E_P\big(|X|\mathbf{1}_A\big) < \varepsilon$$

2. Let $\tau(\omega) \in \mathbb{N}$ be a stopping time w.r.t. $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$. Show that

$$\mathcal{F}_\tau = \big\{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \forall t \in \mathbb{N}\big\}$$

is a σ-algebra.

3. We continue with the random walk. We have

$$M_t(\omega) = \sum_{s=1}^{t} X_s(\omega)$$

is a binary random walk where $t \in \mathbb{N}$ and $(X_s : s \in \mathbb{N})$ are i.i.d. random variables with

$$P(X_s = \pm 1) = P(X_s = \pm 1 | \mathcal{F}_{s-1}) = 1/2$$

X_s is $\mathcal{F}_s$ measurable and P-independent from $\mathcal{F}_{s-1}$.

Recall that $(M_t)_{t\in\mathbb{N}}$ and $(M_t^2 - t)_{t\in\mathbb{N}}$ are $\mathbb{F}$-martingales.

- Consider the stopping time $\tau = \tau_K = \inf\{t : M_t \ge K\}$ for $K \in \mathbb{N}$. Show that $P(\tau < \infty) = 1$.
  Hint: the stopped martingale $(M_{t\wedge\tau} : t \in \mathbb{N})$ is a sub-martingale bounded from above
  (equivalently $(-M_{t\wedge\tau})$ is a supermartingale bounded from below).
  Apply Doob forward convegence theorem,
- Show that P almost surely $M_\tau(\omega) = K$
- Show that $(M_{t\wedge\tau}(\omega) : t \in \mathbb{N})$ is not uniformly integrable.
  Hint: otherwise we could interchange the expectation and the limit for $t \to \infty$ operations.
- Show that $E(\tau) = +\infty$
  Hint: prove it by contradiction, using

$$|M_{t\wedge\tau}(\omega)| \le t \wedge \tau(\omega) \le \tau(\omega) \quad \forall t \in \mathbb{N}$$

Resume : a gambler plays a fair coin-toss game with unit stakes, playing from time 0 until the stopping time $\tau_K(\omega)$, when he quits the game a profit $K > 0$. With probability one $\tau_K(\omega) < \infty$, the gambler always makes a profit K which is arbitrarilty large.

This free-lunch paradox is explained as follows:

The gambler's strategy, to play until $\tau_K(\omega)$ requires an infinite amount of capital, because $\forall M \in \mathbb{N}\ P(\tau_{-M} > \tau_K) > 0$, for any finite amount of capital there is a positive probability to lose everything before τ_K.

And even with an infinite amount of capital at disposal, altough $\tau_K(\omega)$ is P a.s. finite, the expected time for winning K is $E(\tau_K) = \infty$.

4. A three-player ruin problem: Initially, three players have respectively $a, b, c \in \mathbb{N}$ units of capital. Games are independent and each game consists of choosing two players at random and transferring one unit from the first-chosen to the second-chosen player. Once a player is ruined, he is ineligible for further play.

   Let τ_1 be the number of games required for one player to be ruined, and let τ_2 be the number of games required for two players to be ruined.

   Let (X_t, Y_t, Z_t) be the numbers of units possessed by the three players after the t-game, and

$$M_t := X_t Y_t Z_t + \frac{(a+b+c)t}{3} \quad \text{and}$$
$$N_t := X_t Y_t + X_t Z_t + Y_t Z_t + t$$

   - Show that the stopped processes $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ are non-negative $\mathbb{F}$-martingales where $\mathcal{F}_t = \sigma(X_s, Y_s, Z_s, s \leq t)$.
   - Use Doob martingale convergence theorem and Fatou lemma to show that $E(\tau_k) < \infty$, for $k = 1, 2$
   - Knowing that $E(\tau_k) < \infty$, show that $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ are uniformly integrable.
   - Use uniform integrability of the stopped martingales $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ to compute $E(\tau_k)$ for $k = 1, 2$.

5. A generalization of a game by Jacob Bernoulli. In this game a fair die is rolled, and if the result is Z_1, then Z_1 dice are rolled. If the total of the Z_1 dice is Z_2, then Z_2 dice are rolled. If the total of the Z_2 dice is Z_3, then Z_3 dice are rolled, and so on. Let $Z_0 \equiv 1$.

   Find a positive constant α such that

$$M_t(\omega) = Z_t(\omega)\alpha^t \quad t \in \mathbb{N}$$

   is a $\mathbb{F}$-martingale where $\mathcal{F}_t = \sigma(Z_0, Z_1, \ldots, Z_t)$.

   Hint: compute $E(Z_{t+1}|\mathcal{F}_t)$

   What does Doob's martingale convergence theorem tell us about this?

6.
   - If $(M_t(\omega) : t \in \mathbb{N})$ is a $\mathbb{F}$-martingale and $f(x)$ is convex such that $E(|f(X_t)|) < \infty \forall t \in \mathbb{N}$, show that $(f(M_t(\omega)) : t \in \mathbb{N})$ is an $\mathbb{F}$-submartingale.
   - If $(M_t(\omega) : t \in \mathbb{N})$ is a $\mathbb{F}$-submartingale and $f(x)$ is convex non-decreasing such that $E(|f(X_t)|) < \infty \forall t \in \mathbb{N}$, show that $(f(M_t(\omega)) : t \in \mathbb{N})$ is an $\mathbb{F}$-submartingale.

     Hint: use Jensen inequality for conditional expectation.