Stochastic analysis, spring 2013, Exercises-4, 14.02.2013

1. Let $\tau_1(\omega)$ and $\tau_2(\omega)$ stopping times with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in T)$ taking values in T . Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Use the definition of stopping time to show that $\sigma(\omega) = \min(\tau_1(\omega), \tau_2(\omega))$ is a $\mathbb{F}$-stopping time.

2. Let $(M_t : t \in \mathbb{R}^+)$ a $\mathbb{F}$-martingale, and τ a $\mathbb{F}$-stopping time.

   Show that the stopped process $(M_{t\wedge\tau} : t \in \mathbb{R}^+)$

   $$M_t^\tau(\omega) = M_{t\wedge\tau}(\omega) = M_t(\omega)\mathbf{1}(t \le \tau(\omega)) + M_{\tau(\omega)}(\omega)\mathbf{1}(t > \tau(\omega))$$

   is a $\mathbb{F}$-martingale.

   We have shown this when $T = \mathbb{N}$ is discrete by using the martingale transform. In continuous time we have not yet defined such martingale transforms. Prove the statement directly by using the definitions.

3. Let $(M_t(\omega))_{t\in T}$ a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $M_0(\omega) = 0$. Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Define the family of random times $\tau_x : x \in \mathbb{R}$

   $$\tau_x(\omega) = \begin{cases} \inf\{s : M_s \ge x\} & \text{for } x \ge 0 \\ \inf\{s : M_s \le x\} & \text{for } x < 0 \end{cases}$$

   Show that τ_x is a stopping time.

4. Let

   $$M_t(\omega) = \sum_{s=1}^{t} X_s(\omega)$$

   is a binary random walk where $t \in \mathbb{N}$ and $(X_s : s \in \mathbb{N})$ are i.i.d. random variables with

   $$P(X_s = \pm 1) = P(X_s = \pm 1 | \mathcal{F}_{s-1}) = 1/2$$

   X_s is $\mathcal{F}_s$ measurable and P-independent from $\mathcal{F}_{s-1}$.

   - Show that $(M_t)_{t\in\mathbb{N}}$ and $(M_t^2 - t)_{t\in\mathbb{N}}$ are $\mathbb{F}$-martingales.
   - Consider the stopping time $\sigma(\omega) = \min(\tau_a, \tau_b)$ where $a < 0 < b \in \mathbb{N}$, and the stopped martingales $(M_{t\wedge\sigma})_{t\in\mathbb{N}}$ and $(M_{t\wedge\sigma}^2 - t \wedge \sigma)_{t\in\mathbb{N}}$.
     Show that Doob's martingale convergence theorem applies and

     $$\lim_{t\to\infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

     exists P-almost surely.
   - Consider now $(M_{t\wedge\sigma}^2 - t \wedge \sigma)$. Use the martingale property together with the reverse Fatou lemma to show that $E(\sigma) < \infty$ which implies $P(\sigma < \infty) = 1$.
   - For $a < 0 < b \in \mathbb{N}$, compute $P(\tau_a < \tau_b)$.

Hint: a martingale has constant expectation $E_P(M_t) = E_P(M_0)$. This holds also for the stopped martingale $M_t^\tau = M_{t\wedge\tau}$.

5. Let $M_t(\omega) = B_t(\omega)$, $t \in \mathbb{R}^+$, a Brownian motion which is assumed to be $\mathbb{F}$-adapted, and such that for all $0 < s < t$ the increment $(B_t - B_s)$ is P-independent from the σ-algebra $\mathcal{F}_s$.

   Note this since by assumption the Brownian motion is $\mathbb{F}$-adapted, it follows that $\mathcal{F}_t^B = \sigma(B_s : 0 \le s \le t) \subseteq \mathcal{F}_t$, which could be strictly bigger.

   - Show that B_t, $M_t = (B_t^2 - t)$ and $Z_t^a = \exp(aB_t - a^2 t/2)$ are $\mathbb{F}$-martingales.
   - Let $\sigma(\omega) = \min(\tau_a(\omega), \tau_b(\omega))$, for $a < 0 < b \in \mathbb{R}$. We will see in the lectures that the Doob martingale convergence theorem applies also to continuous martingales in continuous time. By following the same line of proof as in the random walk case check that $P(\sigma < \infty) = 1$.
   - Let $a < 0 < b \in \mathbb{R}$. Compute $P(\tau_a < \tau_b)$,

   Hints

   When M is either a Brownian motion or a random walk, the stopped process $M_{t\wedge\sigma}(\omega)$ is a uniformly bounded martingale. To compute $P(\tau_a < \tau_b)$, use first the martingale property

   $$E(M_{t\wedge\sigma}) = E(M_0) = 0,$$

   then for $t \to \infty$ use the bounded convergence theorem.