Stochastic analysis, spring 2013, Exercises-3, 07.02.13

We will construct a continuous Gaussian stochastic process

$$(B(s,t) : s, t \in [0,1])$$

with 2-dimensional index such that

$$E(B(s,t)) = 0 \text{ and } E\big(B(s,t)B(u,v)\big) = (s \wedge u)(t \wedge v)$$

Such process is called the 2-dimensional Brownian sheet or also Wiener-Chentsov random field (this definition extends in any finite dimension).

We introduce the Cameron Martin space of the Brownian sheet

$$H = \left\{ h : [0,1]^2 \to \mathbb{R}, h(0,0) = 0,\ h(s,t) = \int_{[0,t]\times[0,s]} \ddot{h}(x,y)dxdy, \text{ with } \int_0^1 \int_0^1 \big(\ddot{h}(x,y)\big)^2 dxdy < \infty \right\}$$

where $\ddot{h}(x,y) = \partial^2_{xy} h(x,y)$ where we understand that the double derivative exists in weak sense almost everywhere w.r.t. Lebesgue measure and it is a Borel-measurable function.

Consider the dyadics $D = \bigcup_{n\in\mathbb{N}} D_n$ with $D_n = (k2^{-n} : k = 0, 1, \ldots, 2^n)$, and the Haar system $\{\dot{\eta}_d(s) : d \in D\}$ defined in the lecture notes which forms an orthonormal basis in $L^2([0,1], \mathcal{B}, dt)$.

We denote also $E_n = D_n \setminus D_{n-1}$, so that $D = \bigcup_{n\in\mathbb{N}} E_n$ is a disjoint union. At the 0-dyadic level, $\dot{\eta}_0(s) = 0$ and $\dot{\eta}_1(s) = 1$.

It is not difficult to show that the product functions

$$\ddot{\eta}_{d,d'}(s,t) := \dot{\eta}_d(s)\dot{\eta}_{d'}(t)$$

form an orthonormal basis of $L^2([0,1]^2, \mathcal{B}^{\otimes 2}, dt \otimes ds) \simeq L^2([0,1], \mathcal{B}, dt) \otimes L^2([0,1], \mathcal{B}, dt)$.

Define for $s, t \in [0,1]$, $d, d' \in D$,

$$\eta_{d,d'}(s,t) = \int_{[0,s]\times[0,t]} \ddot{\eta}_{d,d'}(x,y)dxdy := \int_0^s \dot{\eta}_d(x)dx \int_0^t \dot{\eta}_{d'}(y)dy = \eta_d(s)\eta_{d'}(s)$$

Given a sequence of i.i.d. standard Gaussian variables $(\xi_{d,d'}(\omega) : d, d' \in D)$, construct the sequence of random continuous functions in H

$$B^{(N)}(s,t,\omega) = \sum_{d\in D_N} \sum_{d'\in D_N} \eta_d(s)\eta_{d'}(t)\xi_{d,d'}(\omega)$$

We show that with probability $P = 1$ this is a Cauchy sequence in the space of continuous functions $C([0,1]^2; \mathbb{R})$ equipped with the supremum norm $|\cdot|_\infty$.

This implies that with probability 1 there is a limit in supremum norm which defines a continuous Gaussian process

$$B(s,t,\omega) = \sum_{d\in D} \sum_{d'\in D} \eta_d(s)\eta_{d'}(t)\xi_{d,d'}(\omega)$$

In order to keep the notation simple, we use the same notation for subsets of pairs of dyadic indexes and the corresponding subspaces of $L^2([0,1]^2, dt^{\otimes 2})$ and by isomorphism also the correponding subspaces of the Cameron Martin space H, which are generated by basis functions corresponding to the index pairs.

For $N \in \mathbb{N}$, let

$$D_N^2 = D_N \otimes D_N = \{\ddot{\eta}_{d,d'} := \dot{\eta}_d \dot{\eta}_{d'} : \quad d, d' \in D_N\}$$

Denote also

$$E_N^{(2)} = D_N^2 \setminus D_{N-1}^2 = E_N \otimes D_{N-1} + D_{N-1} \otimes E_N + E_N \otimes E_N$$

We have the decomposition

$$D^2 = D \otimes D = \bigoplus_{n=0}^{\infty} E_n^{(2)}$$

where $E_n^{(2)}$ are finite and disjoint, and the corresponding finite dimensional subspaces form an orthogonal decomposition of $L^2([0,1]^2, dt^{\otimes 2})$.

At each level $n \in \mathbb{N}$, let

$$\Gamma_n(s,t,\omega) = \sum_{(d,d') \in E_n^{(2)}} \eta_{d,d'}(s,t)\xi_{d,d'}(\omega) = \sum_{(d,d') \in E_n^{(2)}} \eta_d(s)\eta_{d'}(t)\xi_{d,d'}(\omega)$$

Here are the questions :

1. You have to show that the supremum norm is a Radonifying norm for this process, which means

$$\sum_{n=0}^{\infty} P\biggl(\sup_{s,t \in [0,1]} \bigl|\Gamma_n(s,t)\bigr| > n^{-2} \biggr) < \infty$$

   Then, by Borel Cantelli lemma it will follow from this that with probability $P = 1$

$$B^{(N)}(t,s,\omega) = \sum_{n=0}^{N} \Gamma_n(s,t,\omega)$$

   is a Cauchy sequence in $|\cdot|_\infty$ norm.

   To show this basic step, write

$$\Gamma_n(s,t,\omega) = \sum_{(d,d') \in E_n^{(2)}} \eta_d(s)\eta_{d'}(t)\xi_{d,d'}(\omega)$$

$$= \sum_{d,d' \in E_n} \eta_d(s)\eta_{d'}(t)\xi_{d,d'}(\omega) + \sum_{k=0}^{n-1} \sum_{d \in E_n} \sum_{d' \in E_k} \Bigl(\eta_d(s)\eta_{d'}(t)\xi_{d,d'}(\omega) + \eta_{d'}(s)\eta_d(t)\xi_{d',d}(\omega) \Bigr)$$

   and evaluate the supremum of $|\Gamma_n(s,t,\omega)|$ over $s,t \in [0,1]$.

   Note the following facts:

   - we have seen in the one-dimensional case that $\eta_d(s)$ attains its maximum at $s = d$:

$$0 \le \eta_d(s) = \int_0^s \dot{\eta}_d(u)du \le \eta_d(d) = 2^{-(n+1)/2} \text{ for } d \in E_n,$$

- for each k and $d' \neq d'' \in E_k$ $\eta_{d'}$ and $\eta_{d''}$ have disjoint supports.
- for d in E_n and $0 \leq k \leq n$ there is only one $d_k \in E_k$ with $\eta_{d_k}(d) \neq 0$.
- Note function $\Gamma_n(s,t,\omega)$ is piecewise linear which implies

$$\sup_{s,t\in[0,1]} |\Gamma_n(s,t)| = \sup_{s,t\in D_n} |\Gamma_n(s,t)| = \sup_{(d,d')\in E_n^{(2)}} |\Gamma_n(d,d')|$$

Check that

$$\sup_{s,t} |\Gamma_n(s,t)| = \sup_{d,d'\in E_n} \left| \eta_{d'}(d') \left(\sum_{k=0}^{n} \eta_{d_k}(d) \{ \xi_{d_k,d'}(\omega) + \xi_{d',d_k}(\omega) \} \right) \right|$$

where $d_k \in E_k$ is unique at level $k \leq n$ such that $\eta_{d_k}(d) \neq 0$.

Now use the fact that the sum of independent zero mean Gaussian variables is gaussian with zero mean and the variance is the sum of the variances.

2. Show that the limiting process has the postulated covariance structure. Hint show the isometry

$$E_P(B(s,t)^2) = [0,s] \times [0,t]$$

3. In fact the 2-dimensional Brownian sheet has also the α-Hölder continuity property with probability one

$$\sup_{s,t,u,v\in[0,1]} \left\{ \frac{\left|B(s,t,\omega) - B(u,v,\omega)\right|}{\left|(s-u)^2 + (t-v)^2\right|^{\alpha/2}} \right\} \leq K_\alpha(\omega) := \left|B(\cdot,\cdot,\omega)\right|_{C_\alpha} < \infty \,,$$

but for which $\alpha > 0$?

It is possible to proceed as in the one dimensional case. A more direct proof comes by apply Kolmogorov's continuity criterium (in the Lecture notes) which is a general criterium for Hölder continuity using only the covariance structure of the process (not necessarily Gaussian).