Stochastic analysis, spring 2013, Exercises-2, 31.01.13

1. Let $0 < s < t < u$, and $(B_r : r \geq 0)$ a standard Brownian motion with $B_0 = 0$.

   Compute the conditional distribution of B_t conditionally on $\sigma(B_s, B_u)$.

   Hint: Use Bayes' formula.

2. Let $(a_t : t \in \mathbb{N})$, $(b_t : t \in \mathbb{N})$ sequences. Denote $\Delta a_t = a_t - a_{t-1}$ $\Delta b_t = b_t - b_{t-1}$.

   Check Abel's discrete integration by parts formula:

$$a_T b_T = a_0 b_0 + \sum_{t=1}^{T} a_{t-1} \Delta b_t + \sum_{t=1}^{T} b_t \Delta a_t$$

$$= a_0 b_0 + \sum_{t=1}^{T} a_t \Delta b_t + \sum_{t=1}^{T} b_{t-1} \Delta a_t$$

$$= a_0 b_0 + \sum_{t=1}^{T} a_{t-1} \Delta b_t + \sum_{t=1}^{T} b_{t-1} \Delta a_t + \sum_{t=1}^{T} \Delta a_t \Delta b_t$$

3. Let x_t a continuous path with quadratic variation $[x, x]_t$ among the dyadic sequence of partitions Π_n, and a_t a continuous process with finite variation.

   Use Ito formula to show the integration by parts formula.

$$x_t a_t = x_0 a_0 + \int_0^t a_t dx_t + \int_0^t x_s da_s$$

   Hint: consider the function $F(x, a) = xa$.

4. Let x_t a continuous path with quadratic variation $[x, x]_t$ among the dyadic sequence of partitions Π_n, and $z_0 > 0$.

   Show that $z_t = z_0 \exp\big(x_t - \frac{1}{2}[x, x]_t\big)$ satisfies the linear pathwise differential equation

$$dz_t = z_t dx_t,$$

   which is understood in integral sense

$$z_t = z_0 + \int_0^t z_s \overleftarrow{dx_t}$$

5. What is the quadratic variation of z_t ?

6. Show that $z_t^{-1} = z_0^{-1} \exp\big(-x_t + \frac{1}{2}[x, x]_t\big)$ satisfies

$$z_t^{-1} = z_0^{-1} - \int_0^t z_s^{-1} dx_s + \int_0^t z_s^{-1} d[x, x]_s$$

Remarks: note that from the assumptions it follows that z_t is bounded away from zero on any compact interval, which means $1/z_t$ is bounded on compacts.

Note that by definition $[-x, -x]_t = [x, x]_t$.

7. Let a_t be a continuous path with finite first variation, and z_t as before.

   Show that

$$\xi_t = \left(1 + \int_0^t \frac{1}{z_s} da_s\right) z_t$$

   satisfies the linear inhomogeneous pathwise differential equation

$$d\xi_t = \xi_t dx_t + da_t, \quad \xi_0 = z_0$$

8. Let b_t a continuous path with finite first variation and x_t continuous with quadratic variation $[x, x]_t$ among the dyadic sequence of partitions. Show that

$$\int_0^t a_s dx_s = a_t x_t - a_0 x_0 - \int_0^t x_s da_s = \lim_{\Delta(\Pi)\to 0} \sum_{t_i \in \Pi} a_{t_i} \left(x_{t_{i+1}\wedge t} - x_{t_i \wedge t}\right)$$

   it is well defined independently of the sequence of partitions.

   Hint: use Abel discrete integration by parts formula for a partition Π and take limit as $\Delta(\Pi) \to 0$.

9. Show that $y_t = \int\limits_0^t a_s dx_s$ has quadratic variation among the dyadic sequence of partitions given by

$$[y, y]_t = \int_0^t a_s^2 d[x, x]_s$$

   Hint:

$$\left(\int_{t_i}^{t_{i+1}} a_s dx_s\right)^2 = \left(a_{t_i}(x_{t_{i+1}} - x_{t_i}) + \int_{t_i}^{t_{i+1}} (x_{t_{i+1}} - x_s) da_s\right)^2$$

   develop the squares and take sum over $t_i \in \Pi \cap [0, t]$ and let $\Delta(\Pi) \to 0$.