Stochastic analysis, spring 2013, Exercises-14, 16.05.2013

1. Let $(B_t : t \geq 0)$ a Brownian motion which generates the filtration $\mathbb{F} = (\mathcal{F}_t^B)$. Compute the Ito representation of the random variables

   (a)

   $$\int_0^T B_t dt$$

   (b)

   $$\exp\Big(\int_0^T h(t)dB_t\Big)$$

   where $h(t) \in L^2([0,T],dt)$ is deterministic.

   (c) $\sin(B_T)$ and $\cos(B_T)$. Hint: recall that

   $$\exp(iB_t + t/2) = \cos(B_t)\exp(t/2) + i\sin(B_t)\exp(t/2)$$

   is a complex valued martingale.

   Recall also the formula

   $$f(B_T) = E_P\big(f(B_T)\big) + \int_0^T E_P\big(f'(B_T)\big|\mathcal{F}_s^B\big)dB_s = E_P\big(f(B_T)\big) + \int_0^T E_P\bigg(f(B_T)\frac{B_T - B_t}{T-t}\bigg|\mathcal{F}_s^B\bigg)dB_s$$

   Integration by parts is also useful.

2. Prove the following version of Gronwall's lemma: Let $a_t, b_t : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ non-decreasing functions with $a_0 = 0$. If

   $$0 \leq x_t \leq b_t + \int_0^t x_t da_t \forall t \geq 0$$

   then

   $$x_t \leq b_t \exp(a_t)$$

   Hint: to start you can assume first that these are continuous functions.

   Note The following is also true when a_t is continuous non-decreasing and $b \in \mathbb{R}$ is constant,

   $$x_t \leq b + \int_0^t x_s da_s \Longrightarrow x_t \leq b + \int_0^t \exp(a_t - a_s)bds$$

3. • Write the following Ito stochastic integral as a Stratonovich integral plus a process of finite variation

   $$\int_0^t \cos(B_s + W_s)dB_s \tag{1}$$

- Write th semimartingale decomposition of the following Stratonovich integrals

  $$\int_0^t \exp(B_s + W_s) \circ dW_s$$

  where B_t and W_t are independent Brownian motions.

4. Solve the linear Ito stochastic differential equation

   $$X_t^x = x + B_t + \int_0^t \frac{y - X_s^x}{T - s} ds, \quad t \in [0, T]$$

   where B_t is a Brownian motion.

   Write and solve the linear Ito stochastic differential equation for the derivative with respect to the initial value x:

   $$\dot{X}_t = \frac{\partial}{\partial x} X_t^x$$

5. Let $X_t(\omega) \in \mathbb{R}$ a solution of the stochastic differential equation

   $$X_t^x = x + \int_0^t b(s, X_s^x) ds + \int_0^t \sigma(s, X_s^x) dB_s$$

   If $f(x)$ is a smooth test function with bounded derivatives,

   $$f(X_t) - f(x) - \int_0^t (\mathcal{L}_s f)(X_s) ds$$

   where

   $$(\mathcal{L}_s f)(x) = b(s, x) \frac{df}{dx}(x) + \frac{1}{2} \sigma(s, x)^2 \frac{d^2 f}{dx}(x)$$

   - Assume that X_t has density with respect to Lebesgue measure $p(y; s, x)$ at every $s > 0$. Find the partial differential equation satisfied by the density as follows:

     For any smooth test function with compact support $f(x)$ write Ito formula for $f(X_t)$, take expectation and use Fubini to interchange the order of integration with respect to time and probability. Then write the expectations as Lebesgue integrals with densities $p(y; s, x)$.

   - Write also the partial differential equation for the density of the derivative

     $$\dot{X}_t = \frac{\partial}{\partial x} X_t^x$$

     assuming that the density exists.