

Stochastic analysis, spring 2013, Exercises-13, 02.05.2013

1. (Bougerol' identity) Consider

$\sinh(W_t)$ where W_t is Brownian motion and

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2},$$

$$\frac{d}{dx} \sinh(x) = \cosh(x), \quad \frac{d}{dx} \cosh(x) = \sinh(x), \quad \cosh(x)^2 - \sinh(x)^2 = 1$$

Let also B_t and β_t independent Brownian motions, and

$$X_t = e^{B_t} \int_0^t e^{-B_s} d\beta_s \tag{1}$$

Apply Ito's formula to show that

the processes $(X_t : t \geq 0)$ and $(\sinh(W_t) : t \geq 0)$ have the same distributions,

i.e. they satisfy the same stochastic differential equation in Ito sense.

2. We consider a \mathbb{C} -valued continuous local martingale

$$Z_t = X_t + iY_t$$

where X_t and Y_t are \mathbb{R} -valued continuous local martingales in a filtration \mathbb{F} .

- (a) Prove that there is an unique continuous \mathbb{C} -valued process of finite variation $\langle Z, Z \rangle_t$ such that

$$Z_t^2 - \langle Z, Z \rangle_t$$

is a complex local martingale.

- (b) Prove that the following statements are equivalent:

- i. Z_t^2 is a \mathbb{F} -local martingale.
- ii. $\langle Z, Z \rangle_t = 0 \quad \forall t$
- iii. $\langle X, X \rangle = \langle Y, Y \rangle$ and $\langle X, Y \rangle = 0$

Such \mathbb{C} -valued local martingales are called *conformal* local martingales. For example, if B_t and β_t are independent \mathbb{R} -valued Brownian motions in the filtration \mathbb{F} , the \mathbb{C} -valued *planar Brownian motion*

$$W_t = B_t + i\beta_t \tag{2}$$

is a conformal martingale.

- (c) Use Lévy characterization theorem to show that if $Z_t = X_t + iY_t$ is a \mathbb{C} -valued continuous conformal local martingale in a filtration F , there exists a \mathbb{C} -valued planar Brownian motion, such that

$$Z_t = W_{\langle X, X \rangle_t}$$

with

$$W_u = X_{\tau(u)} + iY_{\tau(u)}, \quad \tau(u) = \inf\{t : \langle X, X \rangle_t \geq u\}$$

(d) Show that for a conformal local martingale $Z_t = X_t + iY_t$

$$\langle \Re(Z), \Re(Z) \rangle_t = \frac{1}{2} \langle Z, \bar{Z} \rangle_t \quad (3)$$

(e) Let H_t a \mathbb{C} -valued bounded progressive process, and Z_t a conformal local martingale, Then

$$U_t = \int_0^t H_s dZ_s = \int_0^t \Re(H_s) dX_s - \int_0^t \Im(H_s) dY_s + i \left(\int_0^t \Im(H_s) dX_s + \int_0^t \Re(H_s) dY_s \right)$$

is a conformal martingale with

$$\langle U, \bar{U} \rangle_t = \int_0^t |H_s|^2 d\langle Z, \bar{Z} \rangle_s$$

(f) Let Z_t a continuous conformal local martingale and $F : \mathbb{C} \rightarrow \mathbb{C}$ twice differentiable as function of two real variables, use Ito formula to show

$$F(Z_t) = F(Z_0) + \int_0^t \frac{\partial}{\partial z} F(Z_s) dZ_s + \int_0^t \frac{\partial}{\partial \bar{z}} F(Z_s) d\bar{Z}_s + \frac{1}{4} \int_0^t \Delta F(Z_s) d\langle Z, \bar{Z} \rangle_s$$

where for $z = x + iy$

$$\begin{aligned} \Delta F(z) &= \Delta(F(x, y)) = \frac{\partial^2}{\partial x^2} F(x, y) + \frac{\partial^2}{\partial y^2} F(x, y) \\ &= \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) F(x, y) = \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} F(x, y) \end{aligned}$$

(g) We say that F is harmonic if $\Delta F(z) = 0 \forall z \in \mathbb{C}$. Show that if F is harmonic and Z_t a continuous conformal local martingale, then $F(Z_t)$ is a local martingale.

(h) Let

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

A function $F : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic if and only if

$$\frac{\partial}{\partial \bar{z}} F(z) = 0$$

and in such case we set

$$F'(z) = \frac{\partial}{\partial z} F(z).$$

If $F(z)$ is holomorphic it is necessarily harmonic. Show that in such case, if Z_t a continuous conformal local martingale, then $F(Z_t)$ is a continuous conformal local martingale with

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_s) dZ_s$$

- (i) Show that if W_t is a \mathbb{C} -valued planar Brownian motion and $F : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and non-constant

$$F(W_t) = F(W_0) + \widetilde{W}_{\langle X, X \rangle_t}$$

where $X_t = \Re(F(W_t))$ and

$$\langle X, X \rangle_t = \int_0^t |F'(W_s)|^2 ds$$

and \widetilde{W}_t is a \mathbb{C} -valued planar Brownian motion.

In other words $F(W_t)$ is a time changed Brownian motion. We say that Brownian motion is *conformal invariant*.

Note We can also show that $\langle X, X \rangle_\infty = \infty$.