**Stochastic analysis, spring 2013, Exercises-11, 18.04.2013**

1. (Lenglart's inequality)

   Let $X_t(\omega) \geq 0$ with $X_0 = 0$, and $A_t(\omega) \geq 0$ continuous processes adapted with respect to $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}^+)$, and assume that $A_t$ is non-decreasing such that for all **bounded** stopping times $\tau(\omega)$

   $$E(X_\tau) \leq E(A_\tau) \qquad (1)$$

   We introduce the running maximum $X^*_t(\omega) = \max_{0 \leq s \leq t} X_s(\omega)$

   Prove the following inequalities for **all** $\mathbb{F}$-stopping times $\tau$, (also unbounded ): $\forall \varepsilon, \delta > 0$

   $$\textbf{a)} \qquad P(X^*_\tau > \varepsilon) \leq \frac{E(A_\tau)}{\varepsilon}$$

   $$\textbf{b)} \qquad P(X^*_\tau > \varepsilon, A_\tau \leq \delta) \leq \frac{E(A_\tau \wedge \delta)}{\varepsilon}$$

   $$\textbf{c)} \qquad P(X^*_\tau > \varepsilon) \leq \frac{E(A_\tau \wedge \delta)}{\varepsilon} + P(A_\tau > \delta)$$

   Hint: First assume that $\tau$ is a bounded stopping time, then you can use monotone convergence for unbounded stopping times.

   Define

   $$\sigma(\omega) = \inf\{t : X_t(\omega) > \varepsilon\}$$

   and note that

   $$\{X^*_\tau > \varepsilon\} = \{\sigma < \tau\}$$

   Use the assumption (1) for the stopping time $\sigma \wedge \tau$.

2. Let $M_t$ a continuous $\mathbb{F}$-local martingale. The $\mathbb{F}$-predictable variation $\langle M \rangle_t$ is the non-decreasing process with $\langle M \rangle_0 = 0$ such that

   $$N_t = M_t^2 - \langle M \rangle_t$$

   is a $\mathbb{F}$-local martingale.

   Show that for any $\mathbb{F}$-stopping time $\tau$

   $$P\left( \max_{0 \leq s \leq \tau(\omega)} |M_s(\omega)| > \varepsilon \right) \leq \frac{E(\delta \wedge \langle M \rangle_\tau)}{\varepsilon^2} + P(\langle M \rangle_\tau > \delta)$$

   Hint: use the previous exercise. In order to show that assumption (1) is satisfied, take a localizing sequence for $N_t$ and use Fatou lemma.

3. $\{M_t^{(n)}(\omega)\}_{n \in \mathbb{N}}$ a sequence of $\mathbb{F}$-local martingales and $\tau$ a $\mathbb{F}$-stopping time. Show that as $n \longrightarrow \infty$

   $$\langle M^{(n)} \rangle_\tau \stackrel{P}{\to} 0 \quad \Longrightarrow \quad \max_{0 \leq s \leq \tau} |M_s^{(n)}(\omega)| \stackrel{P}{\to} 0$$

   with convergence in probability.

4. Let $(B_t : t \ge 0)$ a Brownian motion in the filtration $\mathbb{F} = (\mathcal{F}_t : t \ge 0)$.

   (a) Use Ito formula to show that

$$Z_t(\theta) = \exp\left(\theta B_t - \frac{\theta^2}{2} t\right) \tag{2}$$

   is a true martingale.

   Hint: We have shown this already by using the indpendence of the increments from the past. here you are have to check the square integrability condition of an Ito integral.

   (b) Use Ito's formula to compute the semimartingale decomposition of $B_t^n$ for $n \in \mathbb{N}$. into a local martingale and a process of finite variation. Show that the local martingale is a true martingale. Recall that a Gaussian random variable is in $L^p(\Omega)$ $\forall p < \infty$.

   Hint: use Ito's formula and verify the square integrability condition for an Ito integral.

   (c) Compute $E_P(B_t^n)$, for $n \in \mathbb{N}$,by taking expectation in the semi-martingale decomposition.

5. The Hermite polynomials are defined by the Taylor expansion

$$F(x,u) = \exp\big(ux - \frac{u^2}{2}\big) = \sum_{n=0}^{\infty} \frac{u^n}{n!} h_n(x) \tag{3}$$

We see that $h_0(x) = 1$. We rewrite

$$F(x,u) = \exp\left(\frac{x^2}{2} - \frac{(x-u)^2}{2}\right) = e^{x^2/2} \sum_{n=0}^{\infty} \frac{u^n}{n!} \frac{\partial^n}{\partial u^n} \exp\left(-\frac{(t-u)^2}{2}\right)\Bigg|_{u=0}$$

which shows that

$$h_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(\frac{x^2}{2}\right), \quad n \ge 0$$

We also have

$$\begin{aligned} F\big(\frac{x}{\sqrt{a}}, u\sqrt{a}\big) &= \exp\left(u\sqrt{a}\frac{x}{\sqrt{a}} - \frac{u^2 a}{2}\right) \\ &= \exp\left(ux - \frac{u^2 a}{2}\right) = \sum_{n=0}^{\infty} \frac{u^n}{n!} a^{n/2} h_n\big(\frac{x}{\sqrt{a}}\big) = \sum_{n=0}^{\infty} \frac{u^n}{n!} H_n(x,a) \end{aligned}$$

with

$$H_n(x,a) := a^{n/2} h_n\big(\frac{x}{\sqrt{a}}\big). \tag{4}$$

We also set $H_n(x,0) = x^n$.

   (a) Show that

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} H_n(x,a) + \frac{\partial}{\partial a} H_n(x,a) = 0$$

   Hint: use the Taylor expansion above.

(b) Use Ito formula to show when $B_t$ is an $\mathbb{F}$-Brownian motion,

$$H_n(B_t, t) = t^{n/2} h_n\big(B_t/\sqrt{t}\big)$$

is a martingale. Justify the martingale property of the Ito integral.

(c) Use Ito formula to show that when $M_t$ is a continuous local martingale in the $\mathbb{F}$-filtration,

$$H_n(M_t, \langle M\rangle_t) = \langle M\rangle_t^{n/2} h_n\left(\frac{M_t}{\sqrt{\langle M\rangle_t}}\right)$$

is a local martingale.

(d) Show also that

$$H_n(M_t, \langle M\rangle_t) = n! \int_0^t \left(\int_0^{t_1} \dots \int_0^{t_{n-1}} dM_{t_n} dM_{t_{n-1}} \dots \right) dM_{t_1}$$

where on the right we have an **iterated Ito integral** .