Stochastic analysis, spring 2013, Exercises-10, 11.04.2013

1. Consider a Brownian motion $(B_t : t \geq 0)$ in the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, which means that B_t is $\mathbb{F}$-adapted, time continuous, and for each $0 \leq s \leq t$, the conditional distribution of the increment $(B_t - B_s)$ given $\mathcal{F}_s$ is a Gaussian with zero mean and variance $(t-s)$.

   (a) Show that the Brownian motion has the Markov property: $\forall\ s \leq t$ and bounded Borel function $f(x)$,

   $$E_P(f(B_t)|\mathcal{F}_s)(\omega) = E_P(f(B_t)|\sigma(B_s))(\omega) = E_P\biggl(f(x+B_t-B_s)\biggr)\bigg|_{x=B_s(\omega)} =$$
   $$E_P\biggl(f(x+B_{t-s})\biggr)\bigg|_{x=B_s(\omega)} = \varphi(B_s(\omega))$$

   for some bounded Borel-measurable function $\varphi(x)$.

   (b) Show also that for $0 \leq t_0 \leq t_1 \leq \dots \leq t_d$ and $f(x_1,\dots,x_d)$ bounded and Borel measurable,

   $$E_P(f(B_{t_1},\dots,B_{t_d})|\mathcal{F}_{t_0})(\omega) = E_P(f(B_{t_1},\dots,B_{t_d})|\sigma(B_{t_0}))(\omega) =$$
   $$E_P\biggl(f(x+B_{t_1-t_0},\dots,x+B_{t_d-t_0})\biggr)\bigg|_{x=B_{t_0}(\omega)} = \psi(B_{t_0}(\omega))$$

   for some Borel measurable function $\psi(x)$. Hint: use induction.

   (c) Let $\tau(\omega)$ an $\mathbb{F}$-stopping time taking finitely many values. Show first the **strong Markov property** of Brownian motion: for $f(x)$ bounded measurable function,

   $$E_P\bigl(f(B_{\tau+t})\big|\mathcal{F}_\tau\bigr)(\omega) = E_P\bigl(f(B_{\tau+t})\big|\sigma(B_\tau)\bigr)(\omega) =$$
   $$E_P\bigl(f(x+B_t)\bigr)\bigg|_{x=B_\tau(\omega)} = \varphi(B_\tau(\omega)$$

   Hint: remember the definition of stopped σ-algebra $\mathcal{F}_\tau$.

   (d) Show that $(B_{\tau+t} - B_\tau) \overset{P}{\perp\!\!\!\perp} \mathcal{F}_\tau$, and the conditional distribution of $(B_{\tau+t} - B_\tau)$ given $\mathcal{F}_\tau$ is Gaussian with zero mean and variance t. This means that at every stopping time the Brownian motion restarts from the position B_τ independently of the past.

   (e) Show that the strong Markov property for a general $\mathbb{F}$-stopping time τ. Assume that the filtration $\mathbb{F}$ is right continuous. We have shown that there is a sequence of $\mathbb{F}$-stopping times $\tau_n(\omega) \downarrow \tau$ approximating τ from above, with each τ_n taking only finitely many values. Note also that $\mathcal{F}_{\tau_n} \supseteq \mathcal{F}_\tau$.

   (f) Show that if τ is an $\mathbb{F}$-stoppping time $0 \leq t_1 \leq \dots \leq t_d$ and $f(x_1,\dots,x_d)$ bounded and Borel measurable,

   $$E_P\bigl(f(B_{\tau+t_1},\dots,B_{\tau+t_d})\big|\mathcal{F}_\tau\bigr)(\omega) = E_P\bigl(f(B_{\tau+t_1},\dots,B_{\tau+t_d})\big|\sigma(B_{\tau+t_0})\bigr)(\omega) =$$
   $$E_P\bigl(f(x+B_{t_1-t_0},\dots,x+B_{t_d-t_0})\bigr)\bigg|_{x=B_\tau(\omega)} = \psi(B_\tau(\omega))$$

   for some Borel measurable function $\psi(x)$. Hint: use induction.

2. (Reflection prioiciple) Let

$$B_t^* = \max_{0 \le s \le t} B_s$$

the running maximum of Brownian motion.

We show that for $x > 0$, $P(B_t^* > x) = 2P(B_t > x)$.

Consider the stopping time $\tau_x = \inf\{s : B_s > x\}$ and note that $\{B_t^* > x\} = \{\tau_x < t\}$.

Consider the process

$$\widetilde{B}_t = \begin{cases} B_t & t \le \tau_x \\ 2x - B_t & t > \tau_x \end{cases}$$

which is Brownian motion reflected at level t.

(a) Use the strong Markov property to show that $\widetilde{B}_t$ is a Brownian motion in the filtration $\mathbb{F}$.

(b) Note that

$$\{B_t^* > x\} = \{B_t \ge x\} \cup \{\widetilde{B}_t > x\} \tag{1}$$

with $\{B_t \ge x\} \cap \{\widetilde{B}_t > x\} = \emptyset$, and $P(B_t = x) = 0$.
Compute the probability density function of B_t^*.

(c) Compute

$$P(B_t^* > x, B_t > Y)$$

Hint: use (1).

(d) Compute the joint probability density of (B_t^*, B_t).

(e) The running maximum $(B_t^* : t \ge 0)$ is not a Markov process. Show that the pair (B_t^*, B_t) is a strong Markov process, Hint: use the strong Markov property of Brownian motion.

3. The same reflection principle argument holds for a symmetric random walk on $\mathbb{Z}$. Consider a filtration $\mathbb{F} = (\mathcal{F}_n : n \in \mathbb{N})$ in discrete time. Let $(X_n : n \in N)$ an $\mathbb{F}$-adapted process with

$$P(X_n = 1|\mathcal{F}_{n-1}) = P(X_n = -1|\mathcal{F}_{n-1}) = 1/2$$

(which means X_n is indepependent from the past) and

$$S_n = X_1 + \cdots + X_n$$

Note that the proabability law of S_n is the binomial distribution

$$P(S_n = k) = \binom{n}{k} 2^{-n}$$

Let

$$S_n^* = \max_{1 \le k \le n} S_k$$

the running maximum of the random walk.

(a) Show that $(S_n : n \in \mathbb{N})$ is a strong Markov process in the filtration $\mathbb{F}$.

(b) Compute the joint probabiliy $P(S_n^* = \ell, S_n = k)$.

(c) Show that $(S_n^*, S_n)_{n\in\mathbb{N}}$ is a strong Markov process in the filtration $\mathbb{F}$.