Stochastic analysis, spring 2013, Exercises-1, 24.01.13

1. Let $x : \mathbb{R}^+ \to \mathbb{R}$ a function. Its total variation (or first-variation) is defined as

$$v_t(x) = \sup_{\Pi} \sum_{t_i \in \Pi} \big| x_{t_i \wedge t} - x_{t_{i-1} \wedge t} \big| \tag{1}$$

where the supremum is over all finite partitions

$$\Pi = \{0 \le t_0 < t_1 < \dots < t_n\}$$

$t \wedge s = \min\{s, t\}$.

Assume that $s \to x_t$ is right continuous with left limits at all points $s \in [0, t]$. We say that x_s is *cadlag* from the french *Continue a Droite, avec Limite A Gauche*.

We decompose $x_s = x^c_s + \sum_{r \le s} \Delta x_r$ where the path x^c_s is continuous and there are at most countably many jumps, since $\forall n$ the set

$$\big\{ s \in [0, t] : |\Delta x_s| > 1/n \big\} \tag{2}$$

is finite, otherwise there would be an accumulation point, and the function x_s would not have left or right limit at that point.

We can show that

$$v_t(x) = v_t(x^c) + \sum_{r \le t} |\Delta x_r| \tag{3}$$

Show that

$$v_t(x) = \lim_{\Delta(\Pi) \to 0} \sum_{t_i \in \Pi} \big| x_{t_i \wedge t} - x_{t_{i-1} \wedge t} \big|$$

where $\Delta(\Pi) = \max\{|t_i - t_{i-1}| : t_i \in \Pi\}$ and Π is a finite partition. This means that the limits exists for any sequence of partitions (Π_n) with $\Delta(\Pi_n) \to 0$ and the limiting value does not depende on the sequence.

Hint: by triangular inequality, when we refine the partition by adding a point the approximation on the right hand side of (1) does not decrease.

Show the results first for a continuous path (which will be absolutely continuous on the compact interval) .

Assume $v_T(x) < \infty$, for some $T \in (0, \infty]$. For $t \in [0, T]$, let

$$x^{\oplus}_t = \frac{v_t(x) + x_t - x_0}{2}, \quad x^{\ominus}_t = \frac{v_t(x) - x_t + x_0}{2}$$

Show that $x^{\oplus}_t$ and $x^{\ominus}_t$ are non-decreasing satisfying $x^{\oplus}_0 = x^{\ominus}_0 = 0$ and

$$x_t = x_0 + x^{\oplus}_t - x^{\ominus}_t, \quad v_t(x) = x^{\oplus}_t + x^{\ominus}_t \tag{4}$$

Show that if

$$x_t = x_0 + y^{\oplus}_t - y^{\ominus}_t \tag{5}$$

with $y_t^\oplus$ and $y_t^\ominus$ non-decreasing satisfying $y_0^\oplus = y_0^\ominus = 0$, then

$$v_t \le y_t^\oplus + y_t^\ominus$$

Show that the decomposition (4) is minimal among the decompositions (5), meaning that

$$a_t := y_t^\oplus - x_t^\oplus = y_t^\ominus - x_t^\ominus$$

is non-decreasing.

2. Assume that x_t is continuous a path with quadratic variation $[x]_t$ among the sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$, and let a_t a continuous function with finite first variation on compact intervals, that is $v_t(a) < \infty$. Let $F(x,a)$ a $C^{2,1}$ function, with continuous partial derivatives $(x,a) \to F_{xx}(x,a)$ and $(x,a) \to F_a(x,a)$.

   Use Taylor expansion, uniform continuity, and the vague convergence definition of the quadratic variation to show in details the extended Ito-Föllmer formula

$$\begin{aligned} F(x_t,a_t) - F(x_0,a_0) - \int_0^t F_a(x_s,a_s)da_s - \frac{1}{2}\int_0^t F_{xx}(x_s,a_s)d[x]_s = \\ = \int_0^t F_x(x_s)d\overleftarrow{x}_s = \lim_{n\to\infty} \sum_{t_i^n\in\Pi_n} F_x(x_{t_i^n})\big(x_{t_{i+1}^n\wedge t} - x_{t_i^n\wedge t}\big) \end{aligned} \tag{6}$$

   where the last equality defines the pathwise integral.

   Hint: Write the telescopic sum with and use Taylor

$$\begin{aligned} &F(x_{t_{i+1}},a_{t_{i+1}}) - F(x_{t_i},a_{t_i}) = \\ &F(x_{t_{i+1}},a_{t_{i+1}}) - F(x_{t_{i+1}},a_{t_i}) + F(x_{t_{i+1}},a_{t_i}) - F(x_{t_i},a_{t_i}) = \\ &F_a(x_{t_{i+1}},a_{\tau_i})(a_{t_{i+1}} - a_{t_i}) + F_x(x_{t_i},a_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2}F_{xx}(x_{t_i},a_{t_i})(x_{t_{i+1}} - x_{t_i})^2 \\ &+\frac{1}{2}\big(F_{xx}(x_{\sigma_i},a_{t_i}) - F_{xx}(x_{t_i},a_{t_i})\big)(x_{t_{i+1}} - x_{t_i})^2 \end{aligned}$$

   for some $t_i \le \tau_i \le t_{i+1}$, $\quad t_i \le \sigma_i \le t_{i+1}$ which exist by the mid-point theorem of elementary analysis.

3. Assume that x_t is a continuous path with $x_0 = 0$, and quadratic variation $[x]_t = t$, among the dyadic sequence of partitions $\mathcal{D} = \big(t_k^n = k2^{-n} : k \in \mathbb{N}\big)_{n\in\mathbb{N}}$, and let $a_t = \exp(t)$. Use the change of variable formula of classical Riemann-Stieltjes integrals and Ito-Föllmer formula (6) to compute the integral representation of

   - $\sin(a_t)$,
   - $\sin(x_t)$,
   - $\sin(a_t x_t)$.

4. What is the first variation of $\sin(a_t)$?

   What is the quadratic variation of $\sin(x_t)$?

   What is the quadratic variation of $\sin(a_t x_t)$?

   Show that $\sin(x_t)$ and $\sin(a_t x_t)$ have infinite first variation.