Stochastic analysis, 10. exercises

Janne Junnila

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Exercise 1 Consider a Brownian motion $(B_t : t \geq 0)$ in the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, which means that B_t is $\mathbb{F}$-adapted, time continuous, and for each $0 \leq s \leq t$, the conditional distribution of the increment $(B_t - B_s)$ given $\mathcal{F}_s$ is a Gaussian with zero mean and variance $(t-s)$.

(a) Show that the Brownian motion has the Markov property: $\forall s \leq t$ and bounded Borel function $f(x)$,

$$E_P(f(B_t)|\mathcal{F}_s)(\omega) = E_P(f(B_t)|\sigma(B_s))(\omega) = E_P\left(f(x + B_t - B_s)\right)\Big|_{x=B_s(\omega)}$$
$$= E_P\left(f(x + B_{t-s})\right)\Big|_{x=B_s(\omega)} = \varphi(B_s(\omega))$$

for some bounded Borel-measurable function $\varphi(x)$.

(b) Show also that for $0 \leq t_0 \leq t_1 \leq \ldots \leq t_d$ and $f(x_1, \ldots, x_d)$ bounded and Borel measurable,

$$E_P(f(B_{t_1}, \ldots, B_{t_d})|\mathcal{F}_{t_0})(\omega) = E_P(f(B_{t_1}, \ldots, B_{t_d})|\sigma(B_{t_0}))(\omega)$$
$$= E_P(f(x + B_{t_1 - t_0}, \ldots, x + B_{t_d - t_0}))\Big|_{x=B_{t_0}(\omega)} = \psi(B_{t_0}(\omega))$$

for some Borel measurable function $\psi(x)$.

(c) Let $\tau(\omega)$ be an $\mathbb{F}$-stopping time taking finitely many values. Show first the **strong Markov property** of Brownian motion: for $f(x)$ bounded measurable function,

$$E_P(f(B_{\tau+t})|\mathcal{F}_\tau)(\omega) = E_P(f(B_{\tau+t})|\sigma(B_\tau))(\omega) = E_P(f(x + B_t))\Big|_{x=B_\tau(\omega)} = \varphi(B_\tau(\omega)).$$

(d) Show that $(B_{\tau+t} - B_\tau) \perp\!\!\!\perp \mathcal{F}_\tau$, and the conditional distribution of $(B_{\tau+t} - B_\tau)$ given $\mathcal{F}_\tau$ is Gaussian with zero mean and variance t. This means that at every stopping time the Brownian motion restarts from the position B_τ independently of the past.

(e) Show the strong Markov property for a general $\mathbb{F}$-stopping time τ. Assume that the filtration $\mathbb{F}$ is right continuous. We have shown that there is a sequence of $\mathbb{F}$-stopping times $\tau_n(\omega) \downarrow \tau$ approximating τ from above, with each τ_n taking only finitely many values. Note also that $\mathcal{F}_{\tau_n} \supset \mathcal{F}_\tau$.

(f) Show that if τ is an $\mathbb{F}$-stopping time $0 \leq t_0 \leq t_1 \leq \ldots \leq t_d$ and $f(x_1, \ldots, x_d)$ bounded and Borel measurable,

$$E_P(f(B_{\tau+t_1},...,B_{\tau+t_d})|\mathcal{F}_\tau)(\omega) = E_P(f(B_{\tau+t_1},...,B_{\tau+t_d})|\sigma(B_{\tau+t_0}))(\omega)$$
$$= E_P(f(x+B_{t_1-t_0},...,x+B_{t_d-t_0}))\Big|_{x=B_\tau(\omega)} = \psi(B_\tau(\omega))$$

for some Borel measurable function $\psi(x)$.

Solution 1 (a) Because $B_t - B_s$ has conditional distribution $N(0, t-s)$ w.r.t. $\mathcal{F}_s$, B_t has conditional distribution $N(B_s(\omega), t-s)$. Therefore

$$E(f(B_t)|\mathcal{F}_s)(\omega) = \int_{-\infty}^{\infty} f(x)\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-B_s(\omega))^2}{t-s}}\,dx.$$

Similarly for a fixed ω,

$$\begin{aligned} E(f(B_s(\omega)+B_t-B_s)) &= E(E(f(B_s(\omega)+B_t-B_s)|\mathcal{F}_s)) \\ &= E\left(\int_{-\infty}^{\infty} f(B_s(\omega)+x)\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{x^2}{t-s}}\,dx\right) \\ &= E(f(B_t)|\mathcal{F}_s)(\omega) \end{aligned}$$

and

$$E(f(B_s(\omega)+B_{t-s})) = E(E(f(B_s(\omega)+B_{t-s}-B_0)|\mathcal{F}_0)) = E(f(B_t)|\mathcal{F}_s)(\omega).$$

In particular we see that $E(f(B_t)|\mathcal{F}_s)$ is a $\sigma(B_s)$ measurable function, so that

$$E(f(B_t)|\sigma(B_s)) = E(E(f(B_t)|\mathcal{F}_s)|\sigma(B_s)) = E(f(B_t)|\mathcal{F}_s).$$

(b) We have proven the result in the case $d=1$. Assume that we have proven the result for some $d \in \mathbb{N}$. Then

$$\begin{aligned} E(f(B_{t_1},B_{t_2},...,B_{t_{d+1}})|\mathcal{F}_{t_0})(\omega) &= E(E(f(B_{t_1},B_{t_2},...,B_{t_d},B_{t_{d+1}})|\mathcal{F}_{t_d})|\mathcal{F}_{t_0})(\omega) \\ &= E(E(f(B_{t_1}(\omega'),B_{t_2}(\omega'),...,B_{t_d}(\omega'),B_{t_d}(\omega')+B_{t_{d+1}-t_d}))|\mathcal{F}_{t_0})(\omega) \end{aligned}$$

Letting $g(x_1,...,x_d) = E(f(x_1,x_2,...,x_d,x_d+B_{t_{d+1}-t_d}))$ we have by induction that

$$\begin{aligned} E(f(B_{t_1},...,B_{t_{d+1}})|\mathcal{F}_{t_0})(\omega) &= E(g(B_{t_0}(\omega)+B_{t_1-t_0},...,B_{t_0}(\omega)+B_{t_d-t_0})) \\ &= E(f(B_{t_0}(\omega)+B_{t_1-t_0},...,B_{t_0}(\omega)+B_{t_d-t_0},B_{t_0}(\omega)+B_{t_d-t_0}+B_{t_{d+1}-t_d})) \\ &= E(f(B_{t_0}(\omega)+B_{t_1-t_0},...,B_{t_0}(\omega)+B_{t_d-t_0},B_{t_0}(\omega)+B_{t_{d+1}-t_0})) \end{aligned}$$

(c) Assume that $A \in \mathcal{F}_\tau$. We have to show that

$$\int_A f(B_{\tau+t}(\omega))\,dP(\omega) = \int_A E(f(B_\tau(\omega)+B_t))\,dP(\omega).$$

Let $\tau_1, \tau_2, ..., \tau_n$ be the values that τ takes on. Then we can write

$$\int_A f(B_{\tau+t}(\omega))\,dP(\omega) = \sum_{k=1}^{n}\int_{A\cap\{\tau=\tau_k\}} f(B_{\tau+t}(\omega))\,dP(\omega).$$

Now $A \cap \{\tau = \tau_k\} \in \mathcal{F}_{\tau_k}$, so by (a) we get

$$\int_{A\cap\{\tau=\tau_k\}} f(B_{\tau+t}(\omega))\,dP(\omega) = \int_{A\cap\{\tau=\tau_k\}} E(f(B_\tau(\omega)+B_t))\,dP(\omega)$$

and the result follows.

(d) Suppose that $F \in \mathcal{F}_\tau$. Then for any Borel set $A \subset \mathbb{R}$ we have

$$P(F\cap\{\tau=\tau_k\}\cap\{B_{\tau+t}-B_\tau\in A\}) = P(F\cap\{\tau=\tau_k\})P(\{B_{\tau_k+t}-B_{\tau_k}\in A\}).$$

Thus

$$\begin{aligned}
P(F\cap\{B_{\tau+t}-B_\tau\in A\}) &= \sum_{k=1}^n P(F\cap\{\tau=\tau_k\})P(\{B_{\tau_k+t}-B_{\tau_k}\in A\})\\
&= \sum_{k=1}^n P(F\cap\{\tau=\tau_k\})P(\{B_t\in A\})\\
&= P(F)P(\{B_t\in A\}).
\end{aligned}$$

Finally

$$\begin{aligned}
P(\{B_{\tau+t}-B_\tau\in A\}) &= \sum_{k=1}^n P(\{\tau=\tau_k\}\cap\{B_{\tau_k+t}-B_{\tau_k}\in A\})\\
&= \sum_{k=1}^n P(\{\tau=\tau_k\})P(\{B_t\in A\})\\
&= P(\{B_t\in A\}).
\end{aligned}$$

The above calculation shows the independence and that $P(\{B_{\tau+t}-B_\tau\in A\}|\mathcal{F}_\tau) = P(\{B_t\in A\})$, so the conditional distribution is Gaussian with zero mean and variance t.

(e) Let now τ be a general stopping time and τ_n a sequence of stopping times approximating τ from above, with each τ_n taking only finitely many distinct values. Then for all n,

$$E(f(B_{\tau_n+t})|\mathcal{F}_{\tau_n})(\omega) = E(f(B_{\tau_n}(\omega)+B_t)).$$

Since τ_n approximate τ from above, $\mathcal{F}_{\tau_n}\supset\mathcal{F}_\tau$. Thus for any $F\in\mathcal{F}_\tau$,

$$\begin{aligned}
\int_F f(B_{\tau+t}(\omega))\,dP(\omega) &= \lim_{n\to\infty}\int_F f(B_{\tau_n+t}(\omega))\,dP(\omega)\\
&= \lim_{n\to\infty}\int_F E(f(B_{\tau_n+t})|\mathcal{F}_{\tau_n})(\omega)\,dP(\omega)\\
&= \lim_{n\to\infty}\int_F E(f(B_{\tau_n}(\omega)+B_t))\,dP(\omega)\\
&= \int_F E(f(B_\tau(\omega)+B_t))\,dP(\omega).
\end{aligned}$$

This implies that $E(f(B_{\tau+t})|\mathcal{F}_\tau)(\omega) = E(f(B_\tau(\omega)+B_t))$.

(f) Analogous to what we did before...

Exercise 2 Let

$$B_t^* = \max_{0 \le s \le t} B_s$$

be the running maximum of Brownian motion.

We show that for $x > 0$, $P(B_t^* > x) = 2P(B_t > x)$.

Consider the stopping time $\tau_x = \inf\{s : B_s > x\}$ and note that $\{B_t^* > x\} = \{\tau_x < t\}$.

Consider the process

$$\tilde{B}_t = \begin{cases} B_t, & t \le \tau_x \\ 2x - B_t, & t > \tau_x \end{cases}$$

which is Brownian motion reflected at level t.

(a) Use the strong Markov property to show that $\tilde{B}_t$ is a Brownian motion in the filtration $\mathbb{F}$.

(b) Note that

$$\{B_t^* > x\} = \{B_t \ge x\} \cup \{\tilde{B}_t > x\}$$

with $\{B_t \ge x\} \cap \{\tilde{B}_t > x\} = \emptyset$, and $P(B_t = x) = 0$. Compute the probability density function of B_t^*.

(c) Compute

$$P(B_t^* > x, B_t > y).$$

(d) Compute the joint probability density of (B_t^*, B_t).

(e) The running maximum $(B_t^* : t \ge 0)$ is not a Markov process. Show that the pair (B_t^*, B_t) is a strong Markov process.

Solution 2 (a) By the strong Markov property, $W_t = B_{\tau_x + t} - B_{\tau_x} = B_{\tau_x + t} - x$ is a Brownian motion. Similarly $-W_t$ is a Brownian motion. Now

$$\tilde{B}_t = \begin{cases} B_t, & t \le \tau_x, \\ 2x - W_t, & t > \tau_x. \end{cases}$$

This is a Brownian motion since the conditional distribution of $\tilde{B}_t - \tilde{B}_s$ is the same as the conditional distribution of the original process

$$B_t = \begin{cases} B_t, & t \le \tau_x, \\ x + W_t, & t > \tau_x. \end{cases}$$

(b) We have $P(B_t^* > x) = P(B_t > x) + P(\tilde{B}_t > x)$, so

$$P(B_t^* \le x) = 1 - P(B_t > x) - P(\tilde{B}_t > x) = 1 - 2\int_x^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{t}}\, ds.$$

This implies that the probability density function of B_t^* is

$$\frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{t}}.$$

(c)

$$\begin{aligned} P(B_t^* > x, B_t > y) &= P(\{B_t > x\} \cap \{B_t > y\}) + P(\{\tilde{B}_t > x\} \cap \{B_t > y\}) \\ &= P(\{B_t > x \vee y\}) + P(\{\tilde{B}_t > x\} \cap \{2x - \tilde{B}_t > y\}) \\ &= P(\{B_t > x \vee y\}) + P(\{x < \tilde{B}_t < 2x - y\}) \\ &= \int_{x \vee y}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\, ds + \int_{(x, 2x-y)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\, ds. \end{aligned}$$

(d) We wish to find a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_x^{\infty} \int_y^{\infty} f(u, v)\, dv\, du = \int_{x \vee y}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\, ds + \int_{(x, 2x-y)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\, ds. \tag{1}$$

Assume first that $x > y$. By differentiating (1) w.r.t. y, we get that

$$\int_x^{\infty} f(u, y)\, du = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(2x-y)^2}{2t}},$$

and after differentiating this w.r.t. x, we have

$$f(x, y) = \frac{2(2x - y)}{t\sqrt{2\pi t}} e^{-\frac{(2x-y)^2}{2t}}. \tag{2}$$

Assume then that $x < y$. We can write (1) as

$$\int_y^{\infty} \int_x^{v} f(u, v)\, du + \int_v^{\infty} f(u, v)\, du\, dv = \int_y^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\, ds.$$

Now if we use (2), we have

$$\int_v^{\infty} f(u, v)\, du = \frac{1}{\sqrt{2\pi t}} e^{-\frac{v^2}{2t}},$$

so it makes sense to define

$$f(x, y) := \begin{cases} \frac{2(2x-y)}{t\sqrt{2\pi t}} e^{-\frac{(2x-y)^2}{2t}}, & \text{if } x \geq y \\ 0, & \text{if } x \leq y \end{cases}$$

and check that this satisfies (1) for $x > 0, y \in \mathbb{R}$. Indeed

$$
\begin{aligned}
\int_y^\infty \int_x^\infty f(u,v)\,du\,dv &= \int_y^\infty \int_{x\vee v}^\infty f(u,v)\,du\,dv \\
&= \int_y^\infty \int_{x\vee v}^\infty \frac{2(2u-v)}{t\sqrt{2\pi t}} e^{-\frac{(2u-v)^2}{2t}}\,du\,dv \\
&= \int_y^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{(v-2(x\vee v))^2}{2t}}\,dv \\
&= \int_y^{x\vee y} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(v-2x)^2}{2t}}\,dv + \int_{x\vee y}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{v^2}{2t}}\,dv \\
&= \int_{x\vee y}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{v^2}{2t}}\,dv + \int_{y-2x}^{(x\vee y)-2x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\,ds \\
&= \int_{x\vee y}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\,ds + \int_{(x,2x-y)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{s^2}{2t}}\,ds.
\end{aligned}
$$

(e) Let τ be a stopping time. Notice that

$$\mathbf{1}(B^*_{\tau+t} \geq x) = \mathbf{1}(B^*_\tau \geq x) + \mathbf{1}(B^*_\tau < x)\mathbf{1}(\sup_{0\leq s\leq t} B_{\tau+s} \geq x).$$

By the strong Markov property of B_t, we thus see that

$$P(B^*_{\tau+t} \geq x, B_{\tau+t} \geq y|\mathcal{F}_\tau) = P(B^*_{\tau+t} \geq x, B_{\tau+t} \geq y|\sigma(B^*_\tau, B_\tau)).$$

Therefore the conditional distribution of $(B^*_{\tau+t}, B_{\tau+t})$ given $\mathcal{F}_\tau$ is the same as given $\sigma(B^*_\tau, B_\tau)$, which implies the strong Markov property.

Exercise 3 The same reflection principle holds for a symmetric random walk on $\mathbb{Z}$. Consider a filtration $\mathbb{F} = (\mathcal{F}_n : n \in \mathbb{N})$ in discrete time. Let $(X_n : n \in \mathbb{N})$ be an $\mathbb{F}$-adapted process with

$$P(X_n = 1|\mathcal{F}_{n-1}) = P(X_n = -1|\mathcal{F}_{n-1}) = 1/2$$

(which means that X_n is independent from the past) and

$$S_n = X_1 + \ldots + X_n$$

Note that the probability law of S_n is the binomial distribution

$$P(S_n = k) = \binom{n}{k} 2^{-n}$$

Let

$$S^*_n = \max_{1\leq k\leq n} S_k$$

be the running maximum of the random walk.

(a) Show that $(S_n : n \in \mathbb{N})$ is a strong Markov process in the filtration $\mathbb{F}$.

(b) Compute the joint probability $P(S^*_n = \ell, S_n = k)$.

(c) Show that $(S_n^*, S_n)_{n\in\mathbb{N}}$ is a strong Markov process in the filtration $\mathbb{F}$.

Solution 3 (a) We will prove the Markov property. The strong Markov property will then follow the same way as in 1.c. Let $s < t$ and f a bounded measurable function. Then

$$P(S_t - S_s = k|\mathcal{F}_s)(\omega) = P(X_{s+1}+\ldots+X_t = k|\mathcal{F}_s)(\omega) = P(X_1+\ldots+X_{t-s} = k) = P(S_{t-s} = k)$$

by independence. Hence

$$E(f(S_t)|\mathcal{F}_s)(\omega) = \sum_{k=-\infty}^{\infty} f(k + S_s(\omega))P(S_{t-s} = k) = E(f(S_s(\omega) + S_{t-s})).$$

(b) Let $\tilde{S}_n$ be the process reflected after hitting ℓ. Then for $k \leq \ell$ (assuming $n \equiv k$ modulo 2),

$$P(S_n^* \geq \ell, S_n = k) = P(\tilde{S}_n = 2\ell - k) = P(S_n = 2\ell - k) = \binom{n}{2\ell - k}2^{-n}.$$

Thus

$$P(S_n^* = \ell, S_n = k) = P(S_n^* \geq \ell, S_n = k) - P(S_n^* \geq \ell+1, S_n = k) = \binom{n}{2\ell - k}2^{-n} - \binom{n}{2\ell + 2 - k}2^{-n}$$

(c) Completely analogous to 2.e.