Stochastic analysis, 12. exercises

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Let $B_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$ be a 3-dimensional Brownian motion starting from 0 at time 0, with independent components so that $\langle B_t^{(i)}, B_t^{(j)} \rangle_t = \delta_{ij}$.

The process

$$R_t = |B_t| = \sqrt{\sum_{i=1}^{3} (B_t^{(i)})^2}$$

is called the 3-dimensional Bessel process.

Exercise 1 Use Ito formula to compute the semimartingale decomposition of R_t into a continuous local martingale part W_t and a continuous process of finite variation.

Solution 1 Let $f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then by Ito formula

$$\begin{aligned}
R_t &= f(B_t) = f(B_0) + \sum_{i=1}^{3} \int_0^t \frac{\partial f}{\partial x_i}(B_s)\, dB_s^{(i)} + \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(B_s)\, d\langle B^{(i)}, B^{(j)} \rangle_s \\
&= \sum_{i=1}^{3} \int_0^t \frac{B_s^{(i)}}{\sqrt{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2}}\, dB_s^{(i)} + \frac{1}{2} \sum_i \int_0^t \frac{\partial^2 f}{\partial x_i^2}(B_s)\, d\langle B^{(i)} \rangle_s \\
&= \sum_{i=1}^{3} \int_0^t \frac{B_s^{(i)}}{\sqrt{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2}}\, dB_s^{(i)} + \\
&\quad \frac{1}{2} \sum_i \int_0^t \frac{\sqrt{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2} - \frac{(B_s^{(i)})^2}{\sqrt{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2}}}{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2}\, d\langle B^{(i)} \rangle_s \\
&= \sum_{i=1}^{3} \int_0^t \frac{B_s^{(i)}}{\sqrt{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2}}\, dB_s^{(i)} + \int_0^t \frac{1}{\sqrt{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2}}\, ds.
\end{aligned}$$

Thus we have

$$W_t = \sum_{i=1}^{3} \int_0^t \frac{B_s^{(i)}}{\sqrt{(B_s^{(1)})^2 + (B_s^{(2)})^2 + (B_s^{(3)})^2}}\, dB_s^{(i)}$$

as the continuous local martingale part.

Exercise 2 Compute $\langle R\rangle_t = \langle W\rangle_t$ and use Paul Lévy's characterization theorem for Brownian motion to show that the local martingale part of R_t which satisfies

$$W_t = R_t - \int_0^t \frac{1}{R_s}\,ds$$

is a Brownian motion in the filtration $\mathbb{F}$ generated by (B_t).

Solution 2 We have

$$\langle W\rangle_t = \sum_{i=1}^{3}\int_0^t \frac{(B_s^{(i)})^2}{(B_s^{(1)})^2+(B_s^{(2)})^2+(B_s^{(3)})^2}\,d\langle B^{(i)}\rangle_s = t.$$

It follows that W_t is a Brownian motion.

Exercise 3 Show that R_t is a $\mathbb{F}$-submartingale.

Solution 3 By the triangle inequality

$$E(R_t|\mathcal{F}_s) = E(\|B_t\||\mathcal{F}_s) \geq \|E(B_t|\mathcal{F}_s)\| = \|B_s\| = R_s.$$

Let $M_t = R_t^{-1}$ for $t \geq 1$. We start the process at time 1 since $R_0 = 0$.

Exercise 4 Use Ito formula to show that $(M_t)_{t\geq 1}$ is a local martingale, and write its Ito integral representation.

Solution 4 Let $f(t) = \frac{1}{t}$. Then $f'(t) = -\frac{1}{t^2}$ and $f''(t) = \frac{2}{t^3}$. Thus by Ito formula

$$\begin{aligned} M_t &= f(R_t) = f(R_1) - \int_1^t \frac{1}{R_s^2}\,dR_s + \frac{1}{2}\int_1^t \frac{2}{R_s^3}\,d\langle R\rangle_s = f(R_1) - \int_1^t \frac{1}{R_s^2}\,dR_s + \int_1^t \frac{1}{R_s^3}\,ds \\ &= f(R_1) - \int_1^t \frac{1}{R_s^2}\,dW_s - \int_1^t \frac{1}{R_s^3}\,ds + \int_1^t \frac{1}{R_s^3}\,ds \\ &= \frac{1}{R_1} - \int_1^t \frac{1}{R_s^2}\,dW_s \end{aligned}$$

we see that M_t is a local martingale.

Exercise 5 Compute $\langle M\rangle_t$.

Solution 5 We have

$$\langle M\rangle_t = \int_1^t \frac{1}{R_s^4}\,d\langle W\rangle_s = \int_1^t \frac{1}{R_s^4}\,ds.$$

Exercise 6 Show that M_t is a supermartingale.

Solution 6 Let τ_n be a localizing sequence for the local martingale M_t. Because M_t is non-negative, we can use Fatou's lemma to get

$$E(M_t|\,\mathcal{F}_u) = E(\lim_{n\to\infty} M_{t\wedge\tau_n}|\mathcal{F}_u) \leq \liminf_{n\to\infty} E(M_{t\wedge\tau_n}|\mathcal{F}_u) = \liminf_{n\to\infty} M_{u\wedge\tau_n} = M_u.$$

Exercise 7 Let $\tau_a := \inf\{t \geq 1 : R_t \leq a\}, a > 0$, with the convention $\inf\{\emptyset\} = \infty$. Show that the stopped process $(M_t^{\tau_a})_{t\geq 1}$ is a martingale and consequently $(\tau_{1/n} : n \in \mathbb{N})$ is a localizing sequence for the local martingale $(M_t : t \geq 1)$.

Solution 7 Because $M_t^{\tau_a} \leq \frac{1}{a}$, we have by dominated convergence

$$E(M_t^{\tau_a}|\mathcal{F}_s) = E(\lim_{n\to\infty} M_{t\wedge\tau_n}^{\tau_a}|\mathcal{F}_s) = \lim_{n\to\infty} E(M_{t\wedge\tau_n}^{\tau_a}|\mathcal{F}_s) = \lim_{n\to\infty} M_{s\wedge\tau_n}^{\tau_a} = M_s^{\tau_a}$$

for a localizing sequence τ_n.

Exercise 8 Let $0 < r' < y < r''$. Use the martingale property of $(M_{t\wedge\tau_r} : t \geq 1)$ to compute $P(\tau_{r'} < \tau_{r''}|\, R_1 = y)$. By the conditioning we mean that we start R_t at time $t = 1$ in position y.

Solution 8 By Doob optional stopping theorem

$$E(M_{\tau_{r'}\wedge\tau_{r''}}) = \frac{1}{R_1}.$$

Thus we have

$$\frac{1}{y} = P(\tau_{r'} < \tau_{r''}|\, R_1 = y)\frac{1}{r'} + (1 - P(\tau_{r'} < \tau_{r''}|R_1 = y))\frac{1}{r''},$$

from which we can solve

$$P(\tau_{r'} < \tau_{r''}|\, R_1 = y) = \frac{\frac{1}{y} - \frac{1}{r''}}{\frac{1}{r'} - \frac{1}{r''}}.$$

Exercise 9 For $0 < r < y$ compute also $P(\tau_r < \infty|\, R_1 = y)$.

Solution 9 Letting $r'' \to \infty$ in the previous result gives us

$$P(\tau_r < \infty|\, R_1 = y) = \frac{r}{y}.$$

Exercise 10 Show that the 3-dimensional Brownian motion is transient, $|B_t| \to \infty$ P a.s., meaning that it leaves eventually any ball centered around the origin without coming back, and therefore $M_\infty = \lim_{t\to\infty} M_t = 0$.

Solution 10 We play the following game: Starting at $R_1 = y$, we consider the stopping time $\tau_{y/2}$. By 9, there is a probability of $\frac{1}{2}$ that $|B_t| \to \infty$. Otherwise we hit $y/2$ and consider $\tau_{y/4}$. The probability that we will always hit the next smaller ball is 0.

Exercise 11 Using the multivariate Gaussian density in polar coordinates, compute the probability densities of R_t and M_t, and show that the local martingale $(M_t : t \geq 1)$ is bounded in L^2, so that in particular it is uniformly integrable.

Solution 11 The multivariate Gaussian density is

$$\rho_t = \frac{1}{(2\pi t)^{3/2}} e^{-\frac{1}{2}\frac{\|x\|^2}{t}}.$$

Thus

$$P(R_t \le u) = \int_{B(0,u)} \rho_t(x)\,dx = \int_0^u \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{(2\pi t)^{3/2}} e^{-\frac{1}{2}\frac{r^2}{t}} r^2 \cos\phi \,d\phi\,d\theta\,dr$$
$$= \frac{4\pi}{(2\pi t)^{3/2}} \int_0^u e^{-\frac{1}{2}\frac{r^2}{t}} r^2\,dr.$$

It follows that R_t has density

$$P(R_t \in du) = \sqrt{\frac{2}{\pi t^3}} e^{-\frac{1}{2}\frac{u^2}{t}} u^2.$$

Thus

$$P(M_t \le u) = P(R_t \ge \frac{1}{u}) = \int_{\frac{1}{u}}^{\infty} P(R_t \in dv),$$

from which we see that

$$P(M_t \in du) = \sqrt{\frac{2}{\pi t^3}} e^{-\frac{1}{2}\frac{1}{u^2 t}} \frac{1}{u^4}.$$

Assume now $t \ge 1$. Then

$$\int M_t^2\,dP = \int_0^{\infty} u^2 \cdot \sqrt{\frac{2}{\pi t^3}} e^{-\frac{1}{2}\frac{1}{u^2 t}} \frac{1}{u^4}\,du$$
$$= \sqrt{\frac{2}{\pi t^3}} \int_0^{\infty} \frac{e^{-\frac{1}{2}\frac{1}{u^2 t}}}{u^2}\,du = \sqrt{\frac{2}{\pi t^3}} \int_0^{\infty} e^{-\frac{1}{2}\frac{v^2}{t}}\,dv$$
$$= \sqrt{\frac{2}{\pi t^3}} \frac{\sqrt{2\pi t}}{2} = \frac{1}{t}.$$

It follows that M_t is bounded in L^2.

Exercise 12 Compute also the probability density function of R_t^2.

Show first that for $t = 1$,

$$P(R_1^2 \in dx) = \mathbf{1}(x \ge 0) \frac{1}{\Gamma(3/2) 2^{3/2}} \exp(-x/2) x^{\frac{3}{2}-1}\,dx$$

which is the distribution of a Gamma random variable with shape parameter 3/2 and scale parameter 2 (also called chi-square with 3 degrees of freedom) and use the scaling property of Brownian motion.

Solution 12 By the previous exercise, $P(R_t^2 \le u) = P(R_t \le \sqrt{u}) = \int_0^{\sqrt{u}} P(R_t \in dv)$. Therefore the density of R_t^2 is

$$\frac{\sqrt{\frac{2}{\pi t^3}}e^{-\frac{1}{2}\cdot\frac{u}{t}}u}{2\sqrt{u}} = \frac{e^{-\frac{1}{2}\cdot\frac{u}{t}}\sqrt{u}}{\sqrt{2\pi t^3}}$$

Exercise 13 Show that $E(\langle M\rangle_t) = \infty$ for all $t \geq 1$.

Solution 13 By the exercise 5, we have

$$E(\langle M\rangle_t) = \int_1^t \int_0^\infty \frac{1}{x^2} \cdot \frac{e^{-\frac{1}{2}\cdot\frac{x}{s}}\sqrt{x}}{\sqrt{2\pi s^3}}\,dx\,ds = \infty,$$

since the integral of $\frac{1}{x^{3/2}}$ diverges around 0.