Stochastic analysis, 5. exercises

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Exercise 1 We have seen that when $E_P(|X|) < \infty$,

$$\forall \varepsilon > 0, \exists \delta : P(A) < \delta \Longrightarrow E_P(|X|\mathbf{1}_A) < \varepsilon.$$

Show that a collection $\mathcal{C} \subset L^1(P)$ is uniformly integrable if and only if

$$\forall \varepsilon > 0, \exists \delta : P(A) < \delta \Longrightarrow \sup_{X \in \mathcal{C}} E_P(|X|\mathbf{1}_A) < \varepsilon.$$

Solution 1 Assume first that $\mathcal{C}$ is uniformly integrable and let $\varepsilon > 0$ be given. Then (by uniform integrability) there exists $K > 0$ such that

$$\sup_{X \in \mathcal{C}} \int_{\{|X| > K\}} |X| \, dP < \tfrac{\varepsilon}{2}.$$

If we now let $\delta = \frac{\varepsilon}{2K}$, then for any $A \subset \Omega$ with $P(A) < \delta$ we have

$$\begin{aligned}
\sup_{X \in \mathcal{C}} \int_A |X| \, dP &\leq \sup_{X \in \mathcal{C}} \int_{A \cap \{|X| > K\}} |X| \, dP + \sup_{X \in \mathcal{C}} \int_{A \cap \{|X| \leq K\}} |X| \, dP \\
&\leq \sup_{X \in \mathcal{C}} \int_{\{|X| > K\}} |X| \, dP + \sup_{X \in \mathcal{C}} \int_A K \, dP \\
&< \tfrac{\varepsilon}{2} + KP(A) < \varepsilon.
\end{aligned}$$

Suppose then that there exists a constant $L > 0$ such that $E(|X|) < L$ for all $X \in \mathcal{C}$ and the condition

$$\forall \varepsilon > 0, \exists \delta : P(A) < \delta \Longrightarrow \sup_{X \in \mathcal{C}} E_P(|X|\mathbf{1}_A) < \varepsilon.$$

holds. Let $\varepsilon > 0$ and choose δ as in the condition. Then for any $X \in \mathcal{C}$ we have $P(\{|X| > L/\delta\}) < \delta$ and thus

$$\int_{\{|X| > L/\delta\}} |X| \, dP < \varepsilon.$$

Hence

$$\sup_{X \in \mathcal{C}} \int_{\{|X| > L/\delta\}} |X| \, dP < \varepsilon.$$

Because ε was arbitrary, it follows that

$$\lim_{K\to\infty} \sup_{X\in\mathcal{C}} \int_{\{|X|>K\}} |X|\, dP = 0,$$

so $\mathcal{C}$ is uniformly integrable.

Exercise 2 Let $\tau(\omega) \in \mathbb{N}$ be a stopping time w.r.t. $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$. Show that

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{N}\}$$

is a σ-algebra.

Solution 2 Clearly $\emptyset \in \mathcal{F}_\tau$. Suppose that $A \in \mathcal{F}_\tau$. Then for all $t \in \mathbb{N}$

$$\begin{aligned}(\Omega \setminus A) \cap \{\tau \le t\} &= (\Omega \setminus A) \cap (\Omega \setminus \{\tau > t\}) = \Omega \setminus (A \cup \{\tau > t\}) \\ &= \Omega \setminus ((A \cap \{\tau \le t\}) \cup \{\tau > t\}) \in \mathcal{F}_t,\end{aligned}$$

so $\Omega \setminus A \in \mathcal{F}_\tau$. Finally if $A_k \in \mathcal{F}_\tau$ ($k \in \mathbb{N}$) then

$$(\bigcup_{k=1}^{\infty} A_k) \cap \{\tau \le t\} = \bigcup_{k=1}^{\infty} (A_k \cap \{\tau \le t\}) \in \mathcal{F}_t$$

for all $t \in \mathbb{N}$, so $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_\tau$. Thus $\mathcal{F}_\tau$ is a σ-algebra.

Exercise 3 We continue with the random walk. The process

$$M_t(\omega) = \sum_{s=1}^{t} X_s(\omega)$$

is a binary random walk where $t \in \mathbb{N}$ and $(X_s : s \in \mathbb{N})$ are i.i.d. random variables with

$$P(X_s = \pm 1) = P(X_s = \pm 1 | \mathcal{F}_{s-1}) = 1/2.$$

X_s is $\mathcal{F}_s$-measurable and P-independent from $\mathcal{F}_{s-1}$.

Recall that $(M_t)_{t\in\mathbb{N}}$ and $(M_t^2 - t)_{t\in\mathbb{N}}$ are $\mathbb{F}$-martingales.

- Consider the stopping time $\tau = \tau_K = \inf\{t : M_t \ge K\}$ for $K \in \mathbb{N}$. Show that $P(\tau < \infty) = 1$.
- Show that P almost surely $M_\tau(\omega) = K$.
- Show that $(M_{t\wedge\tau}(\omega) : t \in \mathbb{N})$ is not uniformly integrable.
- Show that $E(\tau) = +\infty$.

Solution 3 ($P(\tau < \infty) = 1$ *and* $M_\tau(\omega) = K$): Consider the stopped process $M_{t\wedge\tau}$. It is a martingale and clearly $M_{t\wedge\tau} \le K$ for all $t \in \mathbb{N}$. Thus

$$E(M_{t\wedge\tau}^+) \le K,$$

and by Doob's forward convergence theorem the limit $\lim_{t\to\infty} M_{t\wedge\tau}(\omega)$ exists for almost all ω. Since the limit cannot exist if $\tau(\omega) = \infty$, we must have $P(\tau < \infty) = 1$. If the limit exists, then $M_\tau(\omega) = K$ by the definition of τ.

(*The stopped martingale $M_{t\wedge\tau}$ is not U.I.*): Assume that $M_{t\wedge\tau}$ were U.I. Then $M_{t\wedge\tau} \to M_\tau = K$ in L^1. However

$$E(|M_{t\wedge\tau} - K|) = K - E(M_{t\wedge\tau}) = K,$$

because $M_{t\wedge\tau}$ is a martingale.

($E(\tau) = \infty$): Suppose that $E(\tau) < \infty$. Then because

$$|M_{t\wedge\tau}| \le t \wedge \tau \le \tau,$$

we see that the martingale $M_{t\wedge\tau}$ is uniformly integrable, which is a contradiction.

Exercise 4 A three player ruin problem: Initially, three players have respectively $a, b, c \in \mathbb{N}$ units of capital. Games are independent and each game consists of choosing two players at random and transferring one unit from the first-chosen to the second-chosen player. Once a player is ruined, he is ineligible for further play.

Let τ_1 be the number of games required for one player to be ruined, and let τ_2 be the number of games required for two players to be ruined.

Let (X_t, Y_t, Z_t) be the numbers of units possessed by the three players after the t-game, and

$$M_t := X_t Y_t Z_t + \tfrac{(a+b+c)t}{3}$$
$$N_t := X_t Y_t + Y_t Z_t + Z_t X_t + t$$

- Show that the stopped processes $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ are non-negative $\mathbb{F}$-martingales where $\mathcal{F}_t = \sigma(X_s, Y_s, Z_s, s \le t)$.
- Use Doob martingale convergence theorem and Fatou's lemma to show that $E(\tau_k) < \infty$ for $k = 1, 2$.
- Knowing that $E(\tau_k) < \infty$, show that $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ are uniformly integrable.
- Use uniform integrability of the stopped martingales $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ to compute $E(\tau_k)$ for $k = 1, 2$.

Solution 4 (*The stopped processes are martingales*): Both M_t and N_t are clearly integrable. Moreover, if $\tau_1(\omega) > t$, then

$$\begin{aligned}
E(M_{t+1\wedge\tau_1}|\mathcal{F}_t)(\omega) &= E(X_{t+1\wedge\tau_1} Y_{t+1\wedge\tau_1} Z_{t+1\wedge\tau_1} + \tfrac{(a+b+c)(t+1\wedge\tau_1)}{3}|\mathcal{F}_t)(\omega) \\
&= E(X_{t+1} Y_{t+1} Z_{t+1} + \tfrac{(a+b+c)(t+1)}{3}|\mathcal{F}_t)(\omega) \\
&= \tfrac{1}{6}\Big((X_t - 1)(Y_t + 1)Z_t + X_t(Y_t - 1)(Z_t + 1) + (X_t + 1)Y_t(Z_t - 1) \\
&\quad + (X_t + 1)(Y_t - 1)Z_t + X_t(Y_t + 1)(Z_t - 1) + (X_t - 1)Y_t(Z_t + 1)\Big) \\
&\quad + \tfrac{(a+b+c)(t+1)}{3} \\
&= \tfrac{1}{6}\left(6X_t Y_t Z_t - 2(X_t + Y_t + Z_t)\right) + \tfrac{(a+b+c)(t+1)}{3} = X_t Y_t Z_t + \tfrac{(a+b+c)t}{3}.
\end{aligned}$$

Otherwise if $\tau_1(\omega) \le t$, then clearly $E(M_{t+1\wedge\tau_1}|\mathcal{F}_t)(\omega) = M_{t\wedge\tau_1}$. Consider next $N_{t\wedge\tau_2}$. If $\tau_1(\omega) > t$, then also $\tau_2(\omega) > t$ and

$$\begin{aligned}
E(N_{t+1\wedge\tau_2}|\mathcal{F}_t)(\omega) &= E(X_{t+1}Y_{t+1} + Y_{t+1}Z_{t+1} + Z_{t+1}X_{t+1} + t + 1|\mathcal{F}_t)(\omega) \\
&= \tfrac{1}{6}\Big((X_t+1)(Y_t-1) + (Y_t-1)Z_t + Z_t(X_t+1) \\
&\quad + X_t(Y_t+1) + (Y_t+1)(Z_t-1) + (Z_t-1)X_t \\
&\quad + (X_t-1)Y_t + Y_t(Z_t+1) + (Z_t+1)(X_t-1) \\
&\quad + (X_t-1)(Y_t+1) + (Y_t+1)Z_t + Z_t(X_t-1) \\
&\quad + X_t(Y_t-1) + (Y_t-1)(Z_t+1) + (Z_t+1)X_t \\
&\quad + (X_t+1)Y_t + Y_t(Z_t-1) + (Z_t-1)(X_t+1)\Big) + t + 1 \\
&= \tfrac{1}{6}(6X_tY_t + 6Y_tZ_t + 6Z_tX_t - 6) + t + 1 \\
&= X_tY_t + Y_tZ_t + Z_tX_t + t.
\end{aligned}$$

If $\tau_1(\omega) \le t$ and $\tau_2(\omega) > t$, then exactly one of X_t, Y_t, Z_t is 0. Without loss of generality we can assume that it is X_t.

$$\begin{aligned}
E(N_{t+1\wedge\tau_2}|\mathcal{F}_t)(\omega) &= E(X_{t+1}Y_{t+1} + Y_{t+1}Z_{t+1} + Z_{t+1}X_{t+1} + t + 1|\mathcal{F}_t)(\omega) \\
&= E(Y_{t+1}Z_{t+1} + t + 1|\mathcal{F}_t)(\omega) \\
&= \tfrac{1}{2}\Big((Y_t-1)(Z_t+1) + (Y_t+1)(Z_t-1)\Big) + t + 1 \\
&= Y_tZ_t + t = X_tY_t + Y_tZ_t + Z_tX_t + t.
\end{aligned}$$

Finally if $\tau_2(\omega) \le t$, then clearly $E(N_{t+1\wedge\tau_2}|\mathcal{F}_t) = N_{t\wedge\tau_2}$, so we are done.

(*We have $E(\tau_k) < \infty$*): Because both martingales $M_{t\wedge\tau_1}$ and $N_{t\wedge\tau_2}$ are non-negative, Doob's martingale convergence theorem applies. Therefore

$$M_{t\wedge\tau_1}(\omega) \to M_{\tau_1}(\omega) = \tfrac{(a+b+c)\tau_1(\omega)}{3}$$

for almost every ω. In particular by Fatou's lemma

$$\tfrac{a+b+c}{3}E(\tau_1) = E(\liminf_{t\to\infty} M_{t\wedge\tau_1}) \le \liminf_{t\to\infty} E(M_{t\wedge\tau_1}) = abc,$$

so

$$E(\tau_1) \le \tfrac{3abc}{a+b+c}.$$

Similarly

$$N_{t\wedge\tau_2}(\omega) \to N_{\tau_2}(\omega) = \tau_2(\omega)$$

for almost every ω, so again by Fatou's lemma

$$E(\tau_2(\omega)) = E(\liminf_{t\to\infty} N_{t\wedge\tau_2}) \le \liminf_{t\to\infty} E(N_{t\wedge\tau_2}) = ab + bc + ca.$$

(*The martingales $M_{t\wedge\tau_1}$ and $N_{t\wedge\tau_2}$ are U.I.*): Because

$$M_{t\wedge\tau_1} \le \left(\frac{a+b+c}{3}\right)^3 + \frac{a+b+c}{3}\tau_1,$$

where the right side is integrable, we see that $M_{t\wedge\tau_1}$ is U.I. Similarly

$$N_{t\wedge\tau_2} \le (a+b+c)^2 + \tau_2,$$

and $N_{t\wedge\tau_2}$ is U.I.

(*Evaluating* $E(\tau_k)$): By uniform integrability, we can change the $\le$ signs to $=$ signs above and see that $E(\tau_1) = \frac{3abc}{a+b+c}$ and $E(\tau_2) = ab + bc + ca$.

Exercise 5 A generalization of a game by Jacob Bernoulli. In this game a fair die is rolled, and if the result is Z_1, then Z_1 dice are rolled. If the total of the Z_1 dice is Z_2, then Z_2 dice are rolled. If the total of the Z_2 dice is Z_3, then Z_3 dice are rolled, and so on. Let $Z_0 \equiv 1$.

Find a positive constant α such that

$$M_t(\omega) = Z_t(\omega)\alpha^t, \quad t \in \mathbb{N}$$

is a $\mathbb{F}$-martingale where $\mathcal{F}_t = \sigma(Z_0, Z_1, ..., Z_t)$.

Solution 5 Let $D_t^{(k)}$, $t, k \in \mathbb{N}$ be i.i.d. random variables with $P(D_t^{(k)} = i) = \frac{1}{6}$ for $1 \le i \le 6$. If a constant α exists for which M_t is a martingale, then necessarily

$$\begin{aligned} Z_t\alpha^t &= E(Z_{t+1}\alpha^{t+1}|\mathcal{F}_t) = \alpha^{t+1}E(Z_{t+1}|\mathcal{F}_t) \\ &= \alpha^{t+1}E(\sum_{k=1}^{Z_t} D_{t+1}^{(k)}|\mathcal{F}_t) = \alpha^{t+1}\sum_{k=1}^{Z_t} E(D_{t+1}^{(k)}) \\ &= \alpha^{t+1}Z_t\frac{7}{2}. \end{aligned}$$

Hence $\alpha = \frac{2}{7}$. The above calculation also shows that with this choice of α, M_t is a martingale.

Exercise 6

- If $(M_t(\omega) : t \in \mathbb{N})$ is an $\mathbb{F}$-martingale and $f(x)$ is convex such that $E(|f(X_t)|) < \infty$ $\forall t \in \mathbb{N}$, show that $(f(M_t(\omega)) : t \in \mathbb{N})$ is an $\mathbb{F}$-submartingale.
- If $(M_t(\omega) : t \in \mathbb{N})$ is an $\mathbb{F}$-submartingale and $f(x)$ is convex non-decreasing such that $E(|f(X_t)|) < \infty$ $\forall t \in \mathbb{N}$, show that $(f(M_t(\omega)) : t \in \mathbb{N})$ is an $\mathbb{F}$-submartingale.

Solution 6

- We have

$$E(f(M_{t+1}(\omega))|\mathcal{F}_t) \ge f(E(M_{t+1}(\omega)|\mathcal{F}_t)) = f(M_t).$$

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$$E(f(M_{t+1}(\omega))|\mathcal{F}_t) \ge f(E(M_{t+1}(\omega)|\mathcal{F}_t)) \ge f(M_t).$$