Stochastic analysis, 8. exercises

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Exercise 1 Let $\tau(\omega) \in [0, +\infty]$ be a random time, $F(t) = P(\tau \le t)$ for $t \in [0, \infty)$.

Consider the single jump counting process $N_t = \mathbf{1}(\tau(\omega) \le t)$ which generates the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t^N = \sigma(N_s : s \le t)$.

(a) Show that τ is a stopping time in the filtration $\mathbb{F}$.

(b) Show first that for every Borel function $f(x)$, the random variable

$$f(\tau(\omega))\mathbf{1}(\tau(\omega) \le s)$$

is $\mathcal{F}_s$-measurable.

(c) Define the *cumulative hazard function*

$$\Lambda(t) = \int_0^t \frac{1}{1 - F(s-)} F(ds),$$

where $F(-s) = P(\tau < s)$ denotes the limit from the left. Show that

$$M_t = N_t - \Lambda_{t \wedge \tau}$$

is an $\mathbb{F}$-martingale.

(d) Assume that $t \mapsto F(t)$ and therefore also $t \mapsto \Lambda(t)$ are continuous, which means $P(\tau = t) = 0$ for all $t \in \mathbb{R}^+$. Show that Λ_τ has 1-exponential distribution:

$$P(\Lambda_\tau > x) = \exp(-x), \quad x \ge 0$$

(e) Show that the martingale M_t is uniformly integrable, what is M_∞?

Solution 1 (a) We have $\{\tau(\omega) \le s\} = N_s^{-1}\{1\} \in \mathcal{F}_s^N$, so τ is a stopping time.

(b) We notice that when f is a simple function of the form $f(x) = \sum_{k=1}^n c_n \mathbf{1}(a_n, b_n)$, the claim holds, since

$$f(\tau(\omega))\mathbf{1}(\tau(\omega) \le s) = \sum_{k=1}^n c_n \mathbf{1}(\tau(\omega) \in (a_n, b_n) \cap [0, s])$$

and every set of the form $\{\tau \in (a, b) \cap [0, s]\}$ is $\mathcal{F}_s$ measurable. The rest follows by a standard limiting argument.

(c) Notice first that M_t is integrable, since

$$\int_\Omega \Lambda_{t\wedge\tau}\,dP = \int_\Omega \int_0^{t\wedge\tau} \frac{1}{1-F(s-)} F(ds)\,dP \le \int_0^\infty \int_{\{\tau\ge s\}} \frac{1}{1-F(s-)}\,dP\,F(ds) = \int_0^\infty 1\,F(ds) = 1.$$

We have

$$E(M_t - M_s|\mathcal{F}_s) = E(N_t - \int_0^{t\wedge\tau} \frac{1}{1-F(u-)} F(du) - N_s + \int_0^{s\wedge\tau} \frac{1}{1-F(u-)} F(du)|\mathcal{F}_s) = 0.$$

Thus it is enough to show that for any $A \in \mathcal{F}_s$ we have

$$\int_A \mathbf{1}(s < \tau \le t))\,dP = \int_A \int_{(s\wedge\tau, t\wedge\tau]} \frac{1}{1-F(u-)} F(du)\,dP.$$

Notice that if $\omega \in A$ is such that $\tau(\omega) \le s$, then

$$\mathbf{1}(s < \tau \le t) = 0 = \int_{(s\wedge\tau(\omega), t\wedge\tau(\omega)]} \frac{1}{1-F(u-)} F(du).$$

Therefore we may assume that $\tau(\omega) > s$ for all $\omega \in A$, but this implies that $A = \{\omega \in \Omega : \tau(\omega) > s\}$ (or the empty set, but that is a trivial case). Now

$$\begin{aligned}
\int_A \int_{(s\wedge\tau, t\wedge\tau]} \frac{1}{1-F(u-)} F(du)\,dP &= \int_\Omega \int_{(s, t\wedge\tau]} \frac{\mathbf{1}(\tau(\omega) > s)}{1-F(u-)} F(du)\,dP(\omega) \\
&= \int_{(s,t]} \int_{\{\tau\ge u\}} \frac{1}{1-F(u-)}\,dP\,F(du) \\
&= \int_{(s,t]} \frac{P(\tau \ge u)}{1-P(\tau < u)}\,F(du) \\
&= \int_{(s,t]} F(du) = F(t) - F(s+) = P(\tau \le u) - P(\tau \le s) \\
&= P(s < \tau \le u) = \int_A \mathbf{1}(s < \tau \le t).
\end{aligned}$$

(d) We compute the Laplace transform.

$$\begin{aligned}
\int_\Omega e^{-\theta\Lambda_\tau}\,dP &= \int_\Omega e^{-\theta \int_0^{\tau(\omega)} \frac{1}{1-F(s)} F(ds)}\,dP(\omega) = \int_0^\infty e^{-\theta \int_0^t \frac{1}{1-F(s)} F(ds)}\,F(dt) \\
&= \int_0^\infty e^{\theta \log(1-F(t))} F(dt) = \int_0^\infty (1-F(t))^\theta F(dt) = \frac{1}{\theta+1},
\end{aligned}$$

which is the same as the Laplace transform of a random variable with pdf $\exp(-x)$,

$$\int_0^\infty e^{-\theta x} \exp(-x)\,dx = \int_0^\infty e^{-(\theta+1)x}\,dx = \frac{1}{\theta+1}.$$

(e) The martingale is uniformly integrable, since $M_t \le 1 + \Lambda_\tau$, which is integrable. Moreover, $M_t > 0$, so M_t converges and $E(M_\infty) = E(M_0) = 0$.

Exercise 2 Let $(M_t : t \in \mathbb{R}^+)$ be a $\mathbb{F}$-martingale, and $\mathbb{G}$ a filtration with $\mathcal{G}_t \subset \mathcal{F}_t$. We assume that (M_t) is also $\mathbb{G}$-adapted. Show that (M_t) is a martingale in the smaller filtration $\mathbb{G}$.

Solution 2 We have

$$E(M_t|\mathcal{G}_s) = E((M_t|\mathcal{F}_s)|\mathcal{G}_s) = E(M_s|\mathcal{G}_s) = M_s.$$

Exercise 3 Let $(M_t : t \in \mathbb{R})$ be a $\mathbb{F}$-martingale under P, and $\mathcal{G}_t$ a filtration such that $\forall t \geq 0$, the σ-algebrae $\mathcal{G}_t$ and $\sigma(M_s : s \leq t)$ are P-independent. Show that under P, $(M_t : t \in \mathbb{R}^+)$ is a martingale in the enlarged filtration $(\mathcal{F}_t \vee \mathcal{G}_t : t \geq 0)$.

Solution 3 *Solution missing.*

Let $(B_t : t \geq 0)$ be a Brownian motion in the filtration $\mathbb{F}$, which means

- $B_0(\omega) = 0$
- $t \mapsto B_t(\omega)$ is continuous
- $\forall 0 \leq s \leq t$, $(B_t - B_s)$ is P-independent from $\mathcal{F}_s$, conditionally gaussian with conditional mean $E(B_t - B_s|\mathcal{F}_s) = 0$ and conditional variance $E((B_t - B_s)^2|\mathcal{F}_s) = t - s$.

Exercise 4 Show that for $a > 0$ the process $(a^{-1/2}B_{at} : t \in \mathbb{R}^+)$ is also a Brownian motion.

Solution 4 Notice that it is a Brownian motion w.r.t. the filtration $\mathcal{F}_{at}$!

Clearly at $t = 0$ we have $a^{-1/2}B_{at} = 0$. The continuity is also satisfied. Finally, $a^{-1/2}B_{at} - a^{-1/2}B_{as}$ is clearly independent from $\mathcal{F}_{as}$ and its conditional probability density is just

$$a^{1/2}\frac{1}{\sqrt{2\pi(at-as)}}e^{-\frac{(a^{1/2}x)^2}{2(at-as)}} = \frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{x^2}{2(t-s)}},$$

which is what we needed to prove.

Exercise 5 The process $W_0 = 0$, $W_t = tB_{1/t}$ is also a Brownian motion.

Solution 5 We will show that W_t is a Brownian motion with respect to its own filtration $\mathcal{G}_t = \sigma(W_s : s \leq t)$.

First of all notice that $W_t - W_s = tB_{1/t} - sB_{1/s}$ is independent of $\mathcal{G}_s = \sigma(B_{1/r} : r \leq s)$. We can use the 1st exercise of the 2nd exercise set to see that $B_{1/t}$ has Gaussian conditional distribution with mean $\frac{B_{1/s}/t}{1/s} = \frac{s}{t}B_{1/s}$ and variance $\frac{(1/t)(1/s-1/t)}{1/s} = \frac{t-s}{t^2}$. Thus $tB_{1/t}$ has Gaussian conditional distribution with mean $sB_{1/s}$ and variance $t-s$ and finally $W_t - W_s$ has Gaussian conditional distribution with mean 0 and variance $t-s$, which is what we wanted to show.

Exercise 6 Let $\theta \in \mathbb{R}$, and $i = \sqrt{-1}$ be the imaginary unit.

Show that

$$E(\exp(i\theta B_t)) = \exp\left(-\frac{1}{2}\theta^2 t\right).$$

Solution 6 We calculate

$$\begin{aligned} E(\exp(i\theta B_t)) &= \int_{-\infty}^{\infty} e^{i\theta x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx \\ &= \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2t}} + i\theta\sqrt{\frac{t}{2}}\right)^2 - \frac{\theta^2 t}{2}} \frac{dx}{\sqrt{2\pi t}} \\ &= e^{-\frac{1}{2}\theta^2 t} \frac{1}{\sqrt{\pi}} \int_{(-\infty,\theta\sqrt{\frac{t}{2}})}^{(\infty,\theta\sqrt{\frac{t}{2}})} e^{-z^2}\, dy. \end{aligned}$$

Thus it is enough to show that

$$\int_{(-\infty,a)}^{(\infty,a)} e^{-z^2}\, dz = \sqrt{\pi}$$

for all a. We do this by contour integration. The function e^{-z^2} is analytic, so its integral over the rectangular path $(-R,0) \to (R,0) \to (R,a) \to (-R,a) \to (-R,0)$ is 0. Moreover

$$\int_{(R,0)}^{(R,a)} e^{-z^2}\, dz \to 0, \qquad \int_{(-R,a)}^{(-R,0)} e^{-z^2}\, dz \to 0$$

as $R \to \infty$. Thus letting $R \to \infty$, we get that

$$\int_{(-\infty,a)}^{(\infty,a)} e^{-z^2}\, dz = \int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}.$$

Exercise 7 For $\theta \in \mathbb{R}$, consider now

$$M_t = \exp(i\theta B_t + \frac{1}{2}\theta^2 t) = \left\{\exp(\frac{1}{2}\theta^2 t)\cos(\theta B_t) + \sqrt{-1}\exp(\frac{1}{2}\theta^2 t)\sin(\theta B_t)\right\} \in \mathbb{C}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Recall that $E(\exp(i\theta G)) = \exp(-\theta^2\sigma^2/2)$ when $G(\omega) \sim \mathcal{N}(0,\sigma^2)$.

- Show that M_t is a complex valued $\mathbb{F}$-martingale, which means that real and imaginary parts are $\mathbb{F}$-martingales.
- Show that $\lim_{t\to\infty} |M_t(\omega)| = \infty$.

Solution 7 Assuming $\theta \neq 0$, clearly $\lim_{t\to\infty} |M_t(\omega)| = \lim_{t\to\infty} e^{\frac{1}{2}\theta^2 t} = \infty$.

Now, M_t is integrable since $|M_t| = e^{\frac{1}{2}\theta^2 t} < \infty$. Also the martingale property is satisfied:

$$E(M_t|\mathcal{F}_s) = e^{\frac{1}{2}\theta^2 t + i\theta B_s} E(e^{i\theta(B_t - B_s)}|\mathcal{F}_s) = e^{i\theta B_s + \frac{1}{2}\theta^2 t} e^{-\frac{1}{2}\theta^2(t-s)} = e^{i\theta B_s + \frac{1}{2}\theta^2 s}$$