Stochastic analysis, spring 2013, Exercises-8, 21.03.2013

3 Let $(M_t : t \in \mathbb{R})$ a F-martingale under P, and $\mathcal{G}_t$ a filtration such that $\forall t \geq 0$, the σ-algebrae $\mathcal{G}_t$ and $\sigma(M_s : s \leq t)$ are P-independent.

Show that under P, $(M_t : t \in \mathbb{R}^+)$ is a martingale in the enlarged filtration $(\mathcal{F}_t \vee \mathcal{G}_t : t \geq 0)$.

Solutions This is not always true.

What is true is that $(M_t : t \in \mathbb{R}^+)$ is a martingale in the enlarged filtration $(\sigma(M_s : s \leq t) \vee \mathcal{G} : t \geq 0)$, when the σ-algebra $\mathcal{G}$ is P-independent from $(M_t : t \in \mathbb{R}^+)$

If $G \in \mathcal{G}$ and $A \in \sigma(M_r : r \leq s)$ for $s \leq t$,

$$E_P((M_t - M_s)\mathbf{1}_{A\cap G}) = E_P((M_t - M_s)\mathbf{1}_A \mathbf{1}_G) = E_P((M_t - M_s)\mathbf{1}_A)P(G) = 0$$

and the result follows since

$$\sigma(M_r : r \leq s) \vee \mathcal{G} = \sigma\big(A \cap G : A \in \sigma(M_r : r \leq s), G \in \mathcal{G}\big)$$

Counterexample

Let X_1, X_2, X_3 i.i.d. binary variables with $P(X_i = 1) = P(X_i = 0) = 1/2$, and

$$X_4 = (X_1 + X_2 + X_3) \mod 2$$

It follows that the distribution of (X_1, X_2, X_3, X_4) is invariant under permutations of the coordintates

and for each distinct triple $1 \leq i \neq j \neq k \leq 4$ and $a, b, c \in \{0, 1\}$

$$P(X_i = a, X_j = b, X_k = c) = 2^{-3} = P(X_i = a)P(X_j = b)P(X_k = c)$$

The random variables (X_1, X_2, X_3, X_4) are 3-wise independent but are not independent, since any three random variables determine the 4-th.

Not also that $E_P(X_i) = P(X_i = 1) = 1/2$.

Let

$$M_0 = (X_3 - 1/2), \quad M_1 = (X_3 + X_4 - 1),$$
$$\mathcal{F}_0 = \sigma(X_2, X_3) \subseteq \mathcal{F}_1 = \sigma(X_2, X_3, X_4).$$

Now $(M_t : t = 0, 1)$ is a martingale in the filtration $(\mathcal{F}_t : t = 0, 1)$, is not a martingale in the enlarged filtration $(\mathcal{F}_t \vee \sigma(X_1))$, because $M_1 \neq M_0$ are both $\mathcal{F}_0 \vee \sigma(X_1)$ measurable, which means

$$E(M_1|\mathcal{F}_0) = M_1 \neq M_0.$$