Delayed logistic equation (deterministic)

- $\frac{d\mathbf{X}}{d\mathbf{t}} == \beta\, e^{-\alpha\,\tau}\, \mathbf{X}_\tau - \delta\, \mathbf{X} - \frac{\gamma}{2}\, \mathbf{X}^2$ (**delayed logistic**)
- $\overline{\mathbf{X}} == \frac{\mathbf{2}}{\gamma}\left(\beta\, e^{-\alpha\,\tau} - \delta\right) > \mathbf{0}$ (**equilibrium**)

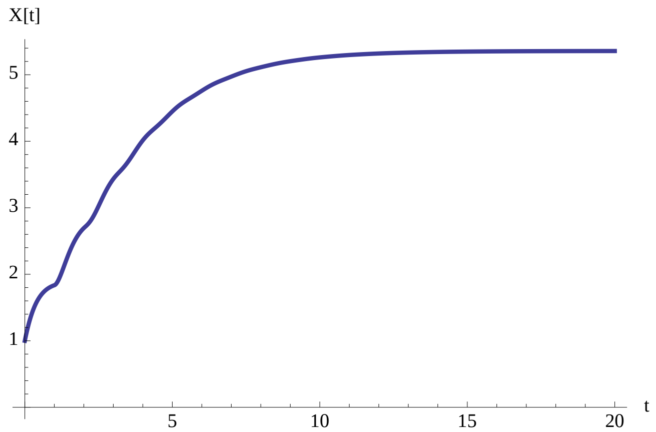
In[1]:=
```
f[X_, Xτ_] := β e^(-α τ) Xτ - δ X - γ/2 X^2;

Xeq := 2/γ (β e^(-α τ) - δ);

α = 1.;
β = 10.;
γ = 1.;
δ = 1.;
τ = 1.;

Print["X̄ = ", Xeq // N]
```

X̄ = 5.35759

In[9]:=
```
sampleDET :=
  Block[{t, X, Xτ, data},
   data = Table[{i Δt, x0}, {i, -n, 0}]; (* initialize data on [-τ,0] *)
   X = x0;
   t = 0;
   While[t < tMax,
    Xτ = data[[-1 - n, 2]];
    X = X + Δt f[X, Xτ];
    t = t + Δt;
    data = Join[data, {{t, X}}];
   ];
   data = Drop[data, n]; (* remove data for [-τ,0] *)
   Return[data];
  ];
```

In[10]:=
```
x0 = 1.; (* initial value on the time interval [-τ,0] *)
n = 100; (* partitioning of the time interval [-τ,0] *)
Δt = τ / n; (* integration time step *)
tMax = 20.; (* stop time *)

seriesDET = sampleDET;

ListPlot[seriesDET, Mesh → False, Joined → True, PlotStyle → Thick,
 AxesOrigin → {0, 0}, AxesLabel → {"t", "X[t]"}, PlotRange → All]
```

Out[15]=



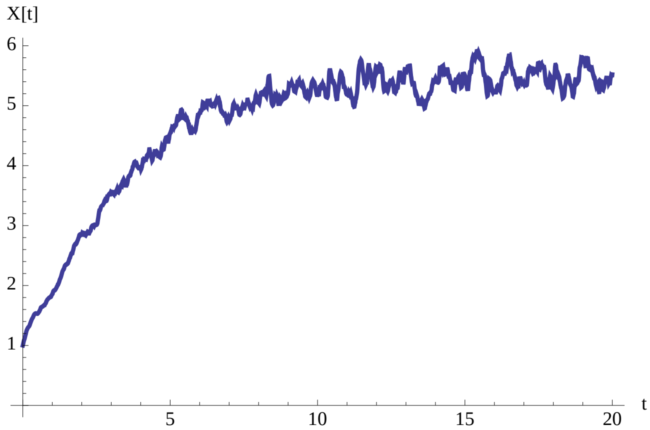
Delayed logistic equation (stochastic)

- $\frac{dX}{dt} == \beta_\tau e^{-\alpha\tau} X_\tau - \delta X - \frac{\gamma}{2} X^2$ (**delayed logistic**)
- $\beta(t) == \beta_0 e^{\zeta(t)}$ (**per capita birth rate**)
- $d\zeta(t) == -a\,\zeta(t)\,dt + b\,dW(t)$ (**Ornstein Uhlenbeck process; $a > 0$**)
- $\zeta(t) \sim N\left(0, \frac{b^2}{2a}\right)$ (**stationary distribution**)
- $\log\beta(t) \sim N\left(\log\beta_0, \frac{b^2}{2a}\right) \Longleftrightarrow \beta(t) \sim \text{logN}\left(\log\beta_0, \frac{b^2}{2a}\right)$ (**stationary distribution**)
- $\overline{\beta} == \beta_0 e^{\frac{b^2}{4a}}$ (**mean birth rate**) $\approx \beta_0\left(1 + \frac{b^2}{4a}\right)$ (**see lecture notes 7.5**)
- $\overline{X} == \frac{2}{\gamma}\left(\beta_0 e^{\frac{b^2}{4a} - \alpha\tau} - \delta\right)$ (**corresponding population density**)

```
In[16]:= f[X_, Xτ_, ξτ_] := β0 ⅇ^(ξτ-α τ) Xτ - δ X - γ/2 X^2;

βmean := β0 ⅇ^(b^2/(4 a));
Xbar := 2/γ (β0 ⅇ^(b^2/(4 a)-α τ) - δ);
α = 1;
β0 = 10;
γ = 1;
δ = 1;
τ = 1;

a = 30;
b = 1;

Print["X̅ = ", Xbar // N]

X̅ = 5.41916

In[27]:= sampleSTO :=
  Block[{t, X, Xτ, ξτ, data},
   data = Table[{i Δt, x0}, {i, -n, 0}]; (* initialize data on [-τ,0] *)
   X = x0;
   ξτ = 0;
   t = 0;
   While[t < tMax,
    Xτ = data[[-1 - n, 2]];
    ξτ = ξτ - a ξτ Δt + b √Δt RandomReal[NormalDistribution[0, 1]];
    X = X + Δt f[X, Xτ, ξτ];
    t = t + Δt;
    data = Join[data, {{t, X}}];
   ];
   data = Drop[data, n]; (* remove data for [-τ,0] *)
   Return[data];
  ];
```

In[28]:=
```
x0 = 1.; (* initial value on the time interval [-τ,0] *)
n = 100; (* partitioning of the time interval [-τ,0] *)
Δt = τ / n; (* integration time step *)
tMax = 20.; (* stop time *)

seriesSTO = sampleSTO;

ListPlot[seriesSTO, Mesh → False, Joined → True, PlotStyle → Thick,
 MaxPlotPoints → Infinity, AxesOrigin → {0, 0}, AxesLabel → {"t", "X[t]"}, PlotRange → All]
```

Out[33]=



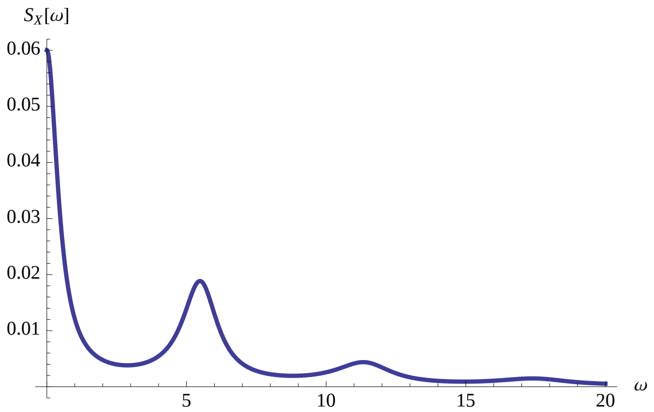
Delayed logistic equation (fluctuation statistics)

- $T(\omega) == \dfrac{\overline{X}\, e^{-\alpha\tau - i\omega\tau}}{i\omega + \delta + \gamma\overline{X} - \overline{\beta}\, e^{-\alpha\tau - i\omega\tau}}$ (**transfer function**)
- $S_\beta(\omega) \approx \dfrac{\beta_0^2\, b^2}{\omega^2 + a^2}$ (**spectral density per capita birth rate; see lecture notes 7.5**)
- $S_X(\omega) \approx (|\mathbf{T}(\omega)|)^2\, S_\beta(\omega)$ (**spectral density population density**)

In[49]:=
```
T[ω_] := Xbar e^(-α τ - i ω τ) / (i ω + δ + γ Xbar - βmean e^(-α τ - i ω τ));

Sβ[ω_] := β0^2 b^2 / (ω^2 + a^2);

SX[ω_] := Abs[T[ω]]^2 Sβ[ω];

CX[t_] := NIntegrate[SX[ω] e^(i ω t), {ω, -30, 30}] / (2 Pi);
```

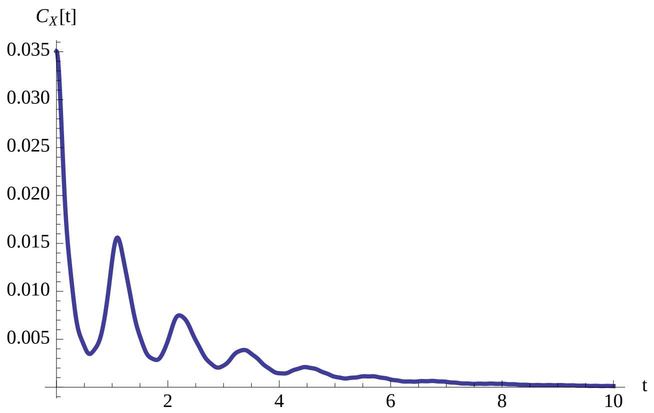
In[53]:=
```
Plot[SX[ω], {ω, 0, 20}, PlotStyle → Thick,
 PlotRange → All, AxesOrigin → {0, 0}, AxesLabel → {"ω", "SX[ω]"}]
```

Out[53]=



In[54]:=
```
Plot[CX[t] // Re, {t, 0, 10}, PlotStyle → Thick,
 PlotRange → All, AxesOrigin → {0, 0}, AxesLabel → {"t", "CX[t]"}]
```

Out[54]=



Delayed logistic equation (along sample path)

- $C_X(t) \;==\; \left\langle \left(x(t'+t) - \overline{X}\right)\left(x(t') - \overline{X}\right)\right\rangle$ **(time-average over sample path)**

$$== \langle x(t'+t)\, x(t')\rangle - \overline{X}^2$$

In[55]:=
```
x0 = 1.; (* initial value on the time interval [-τ,0] *)
n = 100; (* partitioning of the time interval [-τ,0] *)
Δt = τ / n; (* integration time step *)
tTrs = 50; (* length of transient orbit to be ignored chopped off *)
tMax = 500.; (* stop time *)

seriesSTO = Drop[sampleSTO, tTrs / Δt];

Xbar = Mean[seriesSTO[[All, 2]]];

CXnum :=
  Block[{Δn, t, CX, dat},
   dat = {{0, Variance[seriesSTO[[All, 2]]]}};
   Δn = 0;
   While[Δn Δt < 10,
    Δn = Δn + 10;
    t = Δn Δt;
    CX = Mean[Drop[seriesSTO[[All, 2]] - Xbar, Δn] Drop[seriesSTO[[All, 2]] - Xbar, -Δn]];
    dat = Join[dat, {{t, CX}}];
   ];
   Return[dat];
  ];

ListPlot[CXnum, Mesh → False, Joined → True, PlotStyle → Thick,
 PlotRange → All, AxesOrigin → {0, 0}, AxesLabel → {"t", "CX(t)"}]
```

Out[58]=