Numerical integration of SDEs

Ornstein - Uhlenbeck process

- $\begin{cases} dX + a X\, dt == b\, dW \\ X(0) == x_0 \text{ a.s.} \end{cases}$

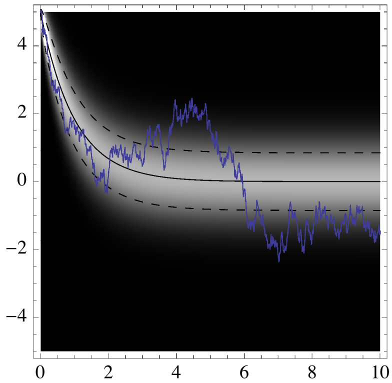
$X(t) \sim \mathrm{N}\left(x_0\, e^{-a t},\ \frac{b^2}{2 a} \left(1 - e^{-2 a t}\right)\right)$

```
a = 1;
b = 1.2;

x0 = 5; (* initial value *)
Δt = .01; (* integration time step *)
tMax = 10; (* max integration time *)

sample :=
  Block[{t, X, data},
   t = 0;
   X = x0;
   data = {{0, x0}};
   While[t < tMax,
    t = t + Δt;
    X = X - a X Δt + b √Δt RandomReal[NormalDistribution[0, 1]];
    data = Join[data, {{t, X}}]];
   Return[data]];
```

```
plot[sample_] :=
  ListPlot[sample, Mesh → False,
   Joined → True, AxesLabel → {"t", "X[t]"}, PlotRange → All];
```

```
background := DensityPlot[PDF[NormalDistribution[x0 ⅇ^(-a t), Sqrt[b^2/(2 a) (1 - ⅇ^(-2 a t))]], x],
   {t, 0, tMax}, {x, -x0, x0}, PlotPoints → 50, ColorFunction → GrayLevel];
std = Plot[{x0 ⅇ^(-a t) + Sqrt[b^2/(2 a) (1 - ⅇ^(-2 a t))], x0 ⅇ^(-a t) - Sqrt[b^2/(2 a) (1 - ⅇ^(-2 a t))], x0 ⅇ^(-a t)},
   {t, 0, tMax}, PlotStyle → {{Dashed, Black}, {Dashed, Black}, {Black}}];
```

```
Show[background, std, plot[sample], ImageSize → Small]
```

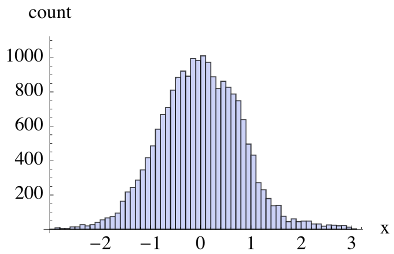



```
a = 1;
b = 1.2;

x0 = 0; (* initial value *)
Δt = .01; (* integration time step *)
tMax = 200; (* max integration time *)

dat = sample[[All, 2]];
mean = Mean[dat];
var = Variance[dat];

Print["Mean = ", mean];
Print["Var = ", var];
Print["b^2/(2a) = ", b^2 / (2 a)];

Histogram[dat, AxesLabel → {"x", "count"}, ImageSize → Small]
```

Mean = -0.00223288

Var = 0.669325

$b^2/(2a)$ = 0.72

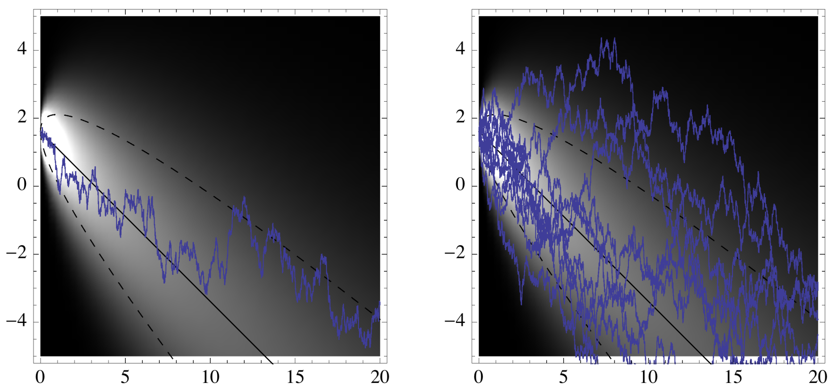


Log-normal process 1

- $\begin{cases} dX == X\, dW \text{ (Ito)} \\ X(0) == x_0 \text{ a.s.} \end{cases}$

$\log(X(t)) \sim \mathrm{N}\left(\log(x_0) - \frac{1}{2} t,\ t\right)$

```
x0 = 5; (* initial value *)
Δt = .01; (* integration time step *)
tMax = 20; (* max integration time *)

sample :=
  Block[{t, X, data},
   t = 0;
   X = x0;
   data = {{0, Log[x0]}};
   While[t < tMax,
    t = t + Δt;
    X = X + X √Δt RandomReal[NormalDistribution[0, 1]];
    data = Join[data, {{t, Log[X]}}]];
   Return[data]];

plot[sample_] :=
  ListPlot[sample, Mesh → False,
   Joined → True, AxesLabel → {"t", "Log x[t]"}, PlotRange → All];

background := DensityPlot[PDF[NormalDistribution[Log[x0] - 1/2 t, √t], x],
   {t, 0, tMax}, {x, -x0, x0}, PlotPoints → 50, ColorFunction → GrayLevel];
std = Plot[{Log[x0] - 1/2 t + √t, Log[x0] - 1/2 t - √t, Log[x0] - 1/2 t},
   {t, 0, tMax}, PlotStyle → {{Dashed, Black}, {Dashed, Black}, {Black}}];

Row[{
  Show[background, std, plot[sample], ImageSize → Small], "     ",
  Show[background, std, Table[plot[sample], {10}], ImageSize → Small]}]
```



Log-normal process 2

- $\begin{cases} dX == X\,dW \text{ (Stratonovitch)} \\ X(0) == x_0 \;\; a.s. \end{cases} \iff \begin{cases} dX == \frac{1}{2} X\,dt + X\,dW \text{ (Ito)} \\ X(0) == x_0 \;\; a.s. \end{cases}$

$\log(X(t)) \sim \mathrm{N}(\log(x_0),\ t)$

```
x0 = 5; (* initial value *)
Δt = .01; (* integration time step *)
tMax = 20; (* max integration time *)

sample :=
  Block[{t, X, data},
   t = 0;
   X = x0;
   data = {{0, Log[x0]}};
   While[t < tMax,
    t = t + Δt;
    X = X + 1/2 X Δt + X Sqrt[Δt] RandomReal[NormalDistribution[0, 1]];
    data = Join[data, {{t, Log[X]}}]];
   Return[data]];

plot[sample_] :=
  ListPlot[sample, Mesh → False,
   Joined → True, AxesLabel → {"t", "Log x[t]"}, PlotRange → All];

background := DensityPlot[PDF[NormalDistribution[Log[x0], Sqrt[t]], x],
   {t, 0, tMax}, {x, -x0, 2 x0}, PlotPoints → 50, ColorFunction → GrayLevel];
std = Plot[{Log[x0] + Sqrt[t], Log[x0] - Sqrt[t], Log[x0]}, {t, 0, tMax},
   PlotStyle → {{Dashed, Black}, {Dashed, Black}, {Black}}];

Row[{
  Show[background, std, plot[sample], ImageSize → Small], "     ",
  Show[background, std, Table[plot[sample], {10}], ImageSize → Small]}]
```