STOCHASTIC POPULATION MODELS

EXERCISES 4-6

4.
Consider the epidemiological model of the last exercise based on the processes

$$\begin{array}{rcll} \mathrm{S}+\mathrm{I} & \xrightarrow{\alpha} & 2\mathrm{I} & \text{(transmission)} \\ \mathrm{I} & \xrightarrow{\beta} & \mathrm{S} & \text{(recovery)} \end{array}$$

Give the transfer function $T(\omega)$ for a seasonally varying infection rate α. Calculate the maximum gain G_{m} and the cutoff frequency ω_{c}. How does the cutoff frequency depend on the average infection rate? Express the cutoff frequency in units of *cycles* per *mean recovery time* (β^{-1}). What does this tell you about absence (or presence) of seasonality of the prevalence of the disease in relation to the average length of an infection?

5.
Consider the resource-consumer model

$$\left\{ \begin{array}{rcll} \frac{\mathrm{d}}{\mathrm{d}t}R & = & R_0 - \alpha R - \beta RC & \text{(resource)} \\ \\ \frac{\mathrm{d}}{\mathrm{d}t}C & = & \gamma\beta RC - \alpha C - \delta C & \text{(consumer)} \end{array} \right.$$

Can you interpret the different terms and parameters? Calculate the equilibria and determine their stability. For the positive equilibrium, give the transfer functions $T_{\mathrm{R}}(\omega)$ and $T_{\mathrm{C}}(\omega)$ for, respectively, the resource and the consumer if the R_0 is seasonally varying. Is there peak frequency where the gain is maximal?

6. Let $\tilde{f}$ and $\tilde{h}$ denote the Fourier transforms of, respectively, f and h, and prove that:

(a) the Fourier transform and its inverse are linear operators,

(b) $\tilde{\tilde{f}}(t) = 2\pi f(-t)$,

(c) $\widetilde{(\frac{d}{dt}f)}(\omega) = i\omega\tilde{f}(\omega)$,

(d) $\frac{d}{d\omega}\tilde{f}(\omega) = -i\widetilde{(tf)}(\omega)$,

(e) $\tilde{f_\tau}(\omega) = e^{-i\omega\tau}\tilde{f}(\omega)$ where $f_\tau(t) := f(t-\tau)$,

(f) $\widetilde{(f*h)}(\omega) = \tilde{f}(\omega)\tilde{h}(\omega)$ where $(f*h)(t) := \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$,

(g) $\widetilde{(fh)}(\omega) = (\tilde{f}*\tilde{h})(\omega)$,

(h) $\int_{-\infty}^{+\infty} f(t)\tilde{h}(t)dt = \int_{-\infty}^{+\infty} \tilde{f}(t)h(t)dt$.