STOCHASTIC POPULATION MODELS

EXERCISES 10-12

10.
Given the stationary stochastic processes $\{X(t)\}$ and $\{Y(t)\}$, show that

(a) $C_X(\tau) = C_X(-\tau)$.

(a) $C_{X,Y}(\tau) = C_{Y,X}(-\tau)$.

(c) $C_{Z,Z} = a^2 C_{X,X} + abC_{X,Y} + abC_{Y,X} + b^2 C_{Y,Y}$.

where the process $\{Z(t)\}$ is defined by $Z(t) := aX(t) + bY(t)$ for given constants a and b.

11.
Given the stationary processes $\{X(t)\}$ and $\{Y(t)\}$, show that

(a) $C_{\frac{\mathrm{d}X}{\mathrm{d}t},Y} = +C'_{X,Y}$

(b) $C_{X,\frac{\mathrm{d}Y}{\mathrm{d}t}} = -C'_{X,Y}$

(c) $C_{\frac{\mathrm{d}X}{\mathrm{d}t},\frac{\mathrm{d}Y}{\mathrm{d}t}} = -C''_{X,Y}$

where $C'_{X,Y}$ and $C''_{X,Y}$ are the first- and second-order derivatives of the cross-covariance function $C_{X,Y}(\tau)$ with respect to τ.

12.
Calculate the spectral density and (if possible) the auto-covariance of the stationary process $\theta(t)$ for each of the following cases:

(a) $\mathrm{d}\theta = -a\theta\,\mathrm{d}t + b\,\mathrm{d}W$

(b) $\mathrm{d}\theta = -a\theta_{\Delta t}\,\mathrm{d}t + b\,\mathrm{d}W$ where $\theta_{\Delta t}(t) := \theta(t - \Delta t)$

(c) $\theta(t) = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} \mathrm{d}W(s)$

(d) $\theta(t) = a\int_{-\infty}^{t} e^{-a(t-s)}\mathrm{d}W(s)$

(e) $\theta(t) = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} \eta(s)\mathrm{d}s$ with $\mathrm{d}\eta = -\eta\,\mathrm{d}t + \mathrm{d}W$

for positive constants a and b.