Appendix A

Local stability analysis

Consider the system in $\mathbb{R}^n$ $(n \geq 1)$

(1) $$\boxed{\frac{dy}{dt} = f(y)}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is cont. diff.
Suppose that $\hat{y}$ is an equilibrium for (1), i.e., $f(\hat{y}) = 0$.

It can be proven that (1) is <u>locally topologically equivalent</u> to the system

(2) $$\boxed{\frac{dx}{dt} = Df(\hat{y})\,x}$$

whenever the eigenvalues of the <u>Jacobi-matrix</u> $Df(\hat{y})$ have non-zero real parts.

(To see the connection between (1) and (2) intuitively, let $x = y - \hat{y}$, and write

$$\frac{d(y-\hat{y})}{dt} = \overbrace{f(\hat{y})}^{=0} + Df(\hat{y})(y-\hat{y}) + \overbrace{o(1)\|y-\hat{y}\|}^{\text{small}}$$

$$\approx Df(\hat{y})(y-\hat{y}).\;)$$

Linear systems

Consider the linear system

(3) $\boxed{\frac{dx}{dt} = Ax}$, $x(0) = x_0$.

where $x \in \mathbb{R}^n$ $(n \geq 1)$, $A \in \mathbb{R}^{n \times n}$.

The formal solution of (3) is

(4) $\boxed{x(t) = e^{At} x_0}$, $t \geq 0$.

where

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k ,$$

which is known to converge for all $A \in \mathbb{R}^{n \times n}$.

But what do the orbits (4) actually look like?

Under what conditions do the orbits converge to zero, the only equilibrium of (3)?

Definition

A matrix $A \in \mathbb{R}^{n\times n}$ is *diagonalizable* if there exist matrices $B, \Lambda \in \mathbb{C}^{n\times n}$ such that $\det B \neq 0$, Λ is diagonal, and $A = B \Lambda B^{-1}$.

$\blacksquare$

Proposition

Suppose that a matrix $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. *Then* the corresponding eigenvectors $b_1, \dots, b_n \in \mathbb{C}^n$ are linearly independent, and

$$A = B \Lambda B^{-1}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $B = (b_1, \dots, b_n)$.

$\blacksquare$

Remark

The proposition not only gives a condition such that a square matrix is diagonalizable, but it also shows that being diagonalizable is the generic situation, because eigenvalues are generically distinct.

$\blacksquare$

Proof of Proposition.

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be distinct eigenvalues of A with corresponding eigenvectors $b_1, \ldots, b_n \in \mathbb{C}^n$.

$\boxed{A b_i = \lambda_i b_i}$ $\forall i$.

Define $B = (b_1, \ldots, b_n)$, then

$\boxed{AB = B\Lambda}$

If we can show that $b_1, \ldots, b_n$ are independent, then it follows immediately that $A = B\Lambda B^{-1}$.

To reach a <u>contradiction</u>, suppose $b_1, \ldots, b_n$ are <u>not</u> independent.

Then there exists an integer $m < n$ such that $b_1, \ldots, b_m$ are independent while $b_1, \ldots, b_m, b_{m+1}$ are not.

Then there exists scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, (not all zeros) such that

$$(*) \quad \boxed{b_{m+1} = \alpha_1 b_1 + \cdots + \alpha_m b_m}$$

Left multiplication with A gives

$$(**) \quad \boxed{\lambda_{m+1} b_{m+1} = \alpha_1 \lambda_1 b_1 + \cdots + \alpha_m \lambda_m b_m}$$

Substitution of $(*)$ into the left-hand-side of $(**)$ and some rearrangement of terms gives

$$\boxed{\alpha_1(\lambda_{m+1}-\lambda_1)b_1 + \dots + \alpha_m(\lambda_{m+1}-\lambda_m)b_m = 0}$$

Since $b_1, \dots, b_m$ are assumed to be independent, it follows that

$\boxed{\alpha_i(\lambda_{m+1}-\lambda_i) = 0}$ for $i = 1, \dots, m$

Since not all α_i are zero, there exists an $i_0 \in \{1, \dots, m\}$ such that $\alpha_{i_0} \neq 0$, and hence $\lambda_{m+1} - \lambda_{i_0} = 0$.
But that contradicts the assumption that $\lambda_1, \dots, \lambda_n$ are all distinct.

∎

Proposition.

Suppose $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and corresponding eigenvectors $b_1, \dots, b_n \in \mathbb{C}^n$.
Then the solution of

$$\boxed{\frac{dx}{dt} = Ax, \quad x(0) = x_0}$$

can be written as

$$(*) \quad \boxed{x(t) = \beta_1 e^{\lambda_1 t} b_1 + \dots + \beta_n e^{\lambda_n t} b_n}$$

for given $\beta_1, \dots, \beta_n \in \mathbb{C}$.

Proof

Let $\beta_1, \dots, \beta_n \in \mathbb{C}$ be such that $x_0 = \beta_1 b_1 + \dots + \beta_n b_n$, which is possible, because by the previous proposition $b_1, \dots, b_n$ form a basis of $\mathbb{C}^n$.

Let $x(t)$ be as in ⊛ on page A5.

Then $x(0) = x_0$, and

$$\frac{dx(t)}{dt} = \lambda_1 \beta_1 e^{\lambda_1 t} b_1 + \dots + \lambda_n \beta_n e^{\lambda_n t} b_n =$$

$$= \beta_1 e^{\lambda_1 t} A b_1 + \dots + \beta_n e^{\lambda_n t} A b_n =$$

$$= A \left(\beta_1 e^{\lambda_1 t} b_1 + \dots + \beta_n e^{\lambda_n t} b_n \right) =$$

$$= A\, x(t).$$

∎

Corollary

Under the conditions of the previous proposition:

① $\mathrm{Re}(\lambda_i) < 0 \ \forall i \implies x(t) \to 0$ as $t \to \infty$.

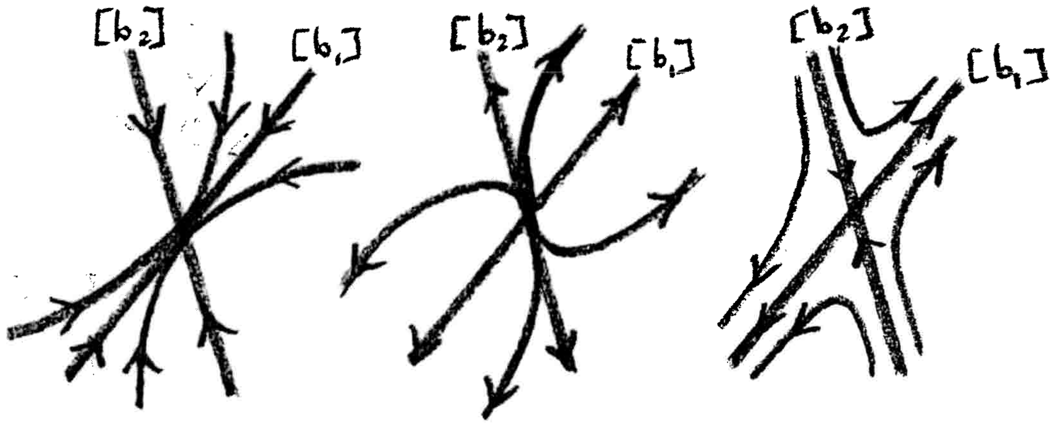
② $\exists\, i \in \{1, \dots, n\} : \mathrm{Re}(\lambda_i) > 0 \implies \|x(t)\| \to \infty$ as $t \to \infty$.

③ $\beta_1 \neq 0,\ \mathrm{Re}(\lambda_1) > \mathrm{Re}(\lambda_i)\ \forall i \neq 1 \implies$
$e^{-\lambda_1 t} x(t) \to \beta_1 b_1$ as $t \to \infty$.

④ Eigenvectors are invariant under $\frac{dx}{dt} = Ax$.

The case $n=2$

λ_1 and λ_2 real and distinct:



$\lambda_2 < \lambda_1 < 0$	$0 < \lambda_2 < \lambda_1$	$\lambda_2 < 0 < \lambda_1$
stable node	unstable node	saddle

λ_1 and λ_2 complex and distinct:

$$A b_1 = \lambda_1 b_1 \Rightarrow A \bar{b}_1 = \bar{\lambda}_1 \bar{b}_1 \quad (A \in \mathbb{R}^{2\times 2})$$

Thus, λ_1 and λ_2 are complex conjugates, and so are the corresponding eigenvectors.

What about the coefficients β_1 and β_2 in (*) on page A5?

Write:

$\lambda_1 = p + iq$, $\lambda_2 = p - iq$, $p, q \in \mathbb{R}$

$b_1 = \begin{pmatrix} u e^{i\varphi} \\ v e^{i\psi} \end{pmatrix}$, $b_2 = \begin{pmatrix} u e^{-i\varphi} \\ v e^{-i\psi} \end{pmatrix}$, $u, v, \varphi, \psi \in \mathbb{R}$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_1, x_2 \in \mathbb{R}$

From ⊛ on page A5:

(**) $$\underbrace{x_1(t)}_{\in \mathbb{R}} = \underbrace{u e^{pt}}_{\in \mathbb{R}} \underbrace{\left[\beta_1 e^{i(qt+\varphi)} + \beta_2 e^{-i(qt+\varphi)}\right]}$$

$\Rightarrow$ must also be real $\forall t$.

- Let $t > 0$ be such that $\boxed{qt + \varphi = 2k\pi}$ for some $k \in \mathbb{N}$
(possible, because $\lambda_1 \neq \lambda_2$, $\lambda_1 = \overline{\lambda_2} \Rightarrow q \neq 0$)

$\Rightarrow$ $\beta_1 + \beta_2 \in \mathbb{R}$ $\Rightarrow$ $\boxed{\operatorname{Im} \beta_1 = - \operatorname{Im} \beta_2}$

- Let $t > 0$ be such that $\boxed{qt + \varphi = \frac{1}{2}\pi + 2k\pi}$ for some $k \in \mathbb{N}$.

$\Rightarrow$ $i\beta_1 - i\beta_2 \in \mathbb{R}$ $\Rightarrow$ $\boxed{\operatorname{Re} \beta_1 = \operatorname{Re} \beta_2}$

Conclusion

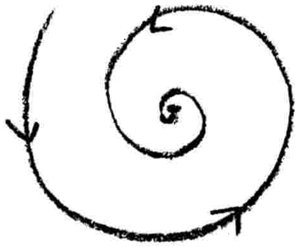
$$\boxed{\beta_1 = \overline{\beta_2}}$$

$= 2u e^{pt} \operatorname{Re}(\mu_1 e^{i(qt+\varphi)}) \underset{\mu_1 = re^{i\mu}}{=} 2u e^{pt} r \operatorname{Re}(e^{i(qt+\varphi+\mu)}) = 2ur e^{pt} \cos(qt+\varphi+\mu)$ ITY

It follows that $x(t)$ is of the following form:

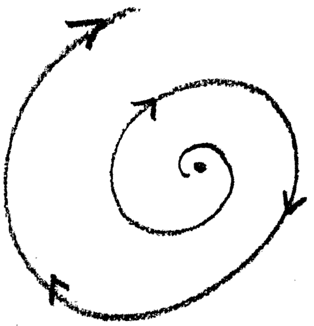
$$\boxed{x(t) = e^{pt}\begin{pmatrix} \mu \cdot \cos(qt+\theta) \\ \nu \cdot \cos(qt+\eta) \end{pmatrix}}$$

for $\mu, \nu, \theta, \eta \in \mathbb{R}$ and where p and q are, respectively, the real and imaginary parts of the eigenvalues of the matrix A.



$p = \operatorname{Re}\lambda < 0$

<u>stable focus</u>



$p = \operatorname{Re}\lambda > 0$

<u>unstable focus</u>

Direction of rotation depends on sign of $q = \operatorname{Im}\lambda$

If

The characteristic equation for $n=2$:

$$\boxed{\lambda^2 - \lambda \,\mathrm{trace}\, A + \det A = 0}$$

Eigenvalues:

$$\boxed{\lambda_{1,2} = \frac{1}{2}\left(\mathrm{trace}\, A \pm \sqrt{(\mathrm{trace}\, A)^2 - 4 \det A}\right)}$$

From the above follow readily the following classification of solutions of $\frac{dx}{dt} = Ax$ in terms of trace A and det A (EXERCISE).