Department of Mathematics and Statistics
Riemannian geometry
Exercise 9
4.4.2013

1. Let $M$ be a Riemannian $n$-manifold and $p \in M$. Show that there exist a neighborhood $U \subset M$ of $p$ and smooth vector fields $E_1, \dots, E_n \in \mathcal{T}(U)$ forming a local orthonormal frame in $U$ such that, at $p$, $\nabla_{E_i} E_j(p) = 0$. [Such a family of vector fields $E_1, \dots, E_n$ is called a geodesic frame at $p$.]

2. Let $A_{ij} : \mathbb{R} \to \mathbb{R}$, $i, j = 1, \dots, n$, be smooth mappings and denote $A = (A_{ij})$. Suppose that $\det A(0) > 0$. Prove that the function $\det A$ has the expansion
$$\frac{\det A(t)}{\det A(0)} = 1 + t \cdot \operatorname{tr}\big(A'A^{-1}\big)(0)$$
$$+ \frac{t^2}{2}\Big(\operatorname{tr}\big(A''A^{-1}\big)(0) - \operatorname{tr}\big((A'A^{-1})^2\big)(0) + \big(\operatorname{tr}\big(A'A^{-1}\big)(0)\big)^2\Big) + O(t^3)$$
in a neighborhood of 0.

3. Prove that in the situation of Exercise 8/5,
$$\det(g_{ij}(\exp_p v)) = 1 - \tfrac{1}{3}\operatorname{Ric}(v, v) + O(|v|^3)$$
for $\exp_p v \in U$.

4. Let $\gamma \colon I \to M$ be a geodesic, $0 \in I$, and $p = \gamma(0)$. Prove that, for every $h \in C^\infty(p)$, we have
$$(h \circ \gamma)''(0) = \operatorname{Hess} h(\dot\gamma_0, \dot\gamma_0).$$

5. Let $M$ be a Riemannian manifold, $p \in U \subset M$, and $R_0 > 0$ such that $\exp_p |B(0, R_0) \colon B(0, R_0) \to U$ is a diffeomorphism. Define $\rho \colon U \to \mathbb{R}$ by setting $\rho(x) = d(x, p)$. Let $\gamma \colon [0, R] \to U$ be a unit speed geodesic, $\gamma(0) = p$, and $0 < R < R_0$. Let $r \in ]0, R]$, $X \in T_{\gamma(r)}M$, $|X| = 1$, $\langle X, \dot\gamma_r\rangle = 0$, and let $\sigma$ be a geodesic such that $\sigma(0) = \gamma(r)$ and $\dot\sigma(0) = X$. Furthermore, let $\Gamma$ be the variation of $\gamma$, where $\Gamma_s$ is the (radial) geodesic from $p$ to $\sigma(s)$. Prove that
$$\operatorname{Hess}\rho(X, X) = \int_0^r \big(|D_t V|^2 - \langle R(V, \dot\gamma)\dot\gamma, V\rangle\big)\, dt,$$
where $V$ is the variation field of $\Gamma$.