Department of Mathematics and Statistics
Riemannian geometry
Exercise 8
21.3.2013

1. Complete the proof of Theorem 6.22: Let $\gamma : [a, b] \to M$ be a geodesic. If $p = \gamma_a$ is not conjugate to $q = \gamma_b$ and $v_1 \in T_pM,\ v_2 \in T_qM$, then there exists a unique Jacobi field V along γ such that $V_a = v_1$ and $V_b = v_2$. (Uniqueness is proven in the lecture notes.)

2. Let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p \colon B(0, \varepsilon) \to B(p, \varepsilon)$ is an isometry whenever $B(p, \varepsilon)$ is a normal ball at p.

3. Let M be complete with $K(\sigma) \leq 0$ for every 2-planes $\sigma \subset T_pM$, $\forall p \in M$. Prove that $\forall p \in M$, $\exp_p : T_pM \to M$ is a local diffeomorphism.

4. Let M be a Riemannian manifold, $p \in M,\ x, y, z \in T_pM,\ |x| = 1$ and $\gamma = \gamma^x$. Let Y and Z be Jacobi fields along γ such that $Y_0 = 0,\ Y_0' = (D_tY)_0 = y,\ Z_0 = 0$, and $Z_0' = (D_tZ)_0 = z$. Prove that
$$\langle Y_t, Z_t\rangle = t^2\langle y, z\rangle - \tfrac{t^4}{3}\langle R(y, x)x, z\rangle + O(t^5).$$

5. Let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM, (U, φ) the corresponding normal chart at p, and g_{ij} the corresponding component functions of the Riemannian metric. Prove that
$$g_{ij}(\exp_p v) = \delta_{ij} - \tfrac{1}{3}\langle R(e_i, v)v, e_j\rangle + O(|v|^3),$$
for $\exp_p v \in U$.