Department of Mathematics and Statistics
Riemannian geometry
Exercise 6
28.2.2013

1. Suppose that N and N' are submanifolds of M and that $\gamma : [0, d] \to M$ is a unit speed geodesic such that $\gamma(0) \in N$, $\gamma(d) \in N'$, and that $\ell(\gamma) = d = d(N, N') > 0$. Here $d(N, N') = \inf\{d(x, y) : x \in N, y \in N'\}$. (In other words, γ minimizes the distance between N and N'.)
   Show that $\dot\gamma_0 \perp T_{\gamma(0)}N$ and $\dot\gamma_d \perp T_{\gamma(d)}N'$. [Use Exercise 5/5]

2. Prove the following version of the Gauss lemma: Let $p \in M$ and $v \in T_pM$ a vector such that $\exp_p v$ is defined. Let $w \in T_v(T_pM) = T_pM$. Then
$$\langle \exp_{p*v}(v), \exp_{p*v}(w)\rangle = \langle v, w\rangle.$$

3. Show that any connected Riemannian manifold (M, g) admits a Riemannian metric $\tilde g = \varphi g$, where $\varphi\colon M \to \mathbb{R}$ is a positive C^∞-function, such that $(M, \tilde g)$ is bounded. In other words, there exists a constant C such that $d_{\tilde g}(x, y) \le C$ for all $x, y \in M$.
   [Hint: The following facts may be useful. (a): If $h\colon M \to \mathbb{R}$ is a non-negative continuous function, then there exists a C^∞-function $f\colon M \to \mathbb{R}$ s.t. $f(x) > h(x)$ for all $x \in M$. (b): For every $\varepsilon > 0$ and for every $p, q \in M$, there exists an admissible path $\gamma\colon [0, L] \to M$ such that $L = \ell(\gamma) \le d(p, q) + \varepsilon$ and $|\dot\gamma_t| = 1$ except for finitely many $t \in [0, L]$.]

4. Let M and N be Riemannian manifolds and $f\colon M \to N$ a diffeomorphism. Suppose that N is complete and that there exists a constant $c > 0$ such that
$$|v| \ge c|f_{*p}v|$$
   for all $p \in M$ and for all $v \in T_pM$. Prove that M is complete.

5. Let M be a complete connected Riemannian manifold, N a Riemannian manifold and $f : M \to N$ a smooth mapping that is a local isometry. Suppose that for every $x, y \in N$ there exists a unique geodesic from x to y. Prove that f is bijective (and hence an isometry).
   [You may use the fact that local isometries preserve geodesics.]