Department of Mathematics and Statistics
Riemannian geometry
Exercise 5
21.2.2013

1. Let M and $\tilde{M}$ be Riemannian manifolds and $\varphi\colon M \to \tilde{M}$ an isometry. Prove that for every $p \in M$ the diagram

$$\begin{array}{ccc} T_pM & \xrightarrow{\varphi_*} & T_{\varphi(p)}\tilde{M} \\ \exp_p \downarrow & & \downarrow \exp_{\varphi(p)} \\ M & \xrightarrow{\varphi} & \tilde{M} \end{array}$$

commutes.

2. Let M be a connected Riemannian manifold and $d : M \times M \to \mathbb{R}$ the metric defined by the Riemannian metric as in Theorem 4.24. Fix $p \in M$ and $L > 1$. Let $\varphi : U \to \varphi U$ be a normal chart at p. Prove that there exists $r > 0$ and a normal ball $B = \exp_p\big(B(0,r)\big) \subset U$ such that $\varphi|B : B \to \varphi B$ is L-bilipschitz from the metric space (B, d) to $\varphi B \subset \mathbb{R}^n$, i.e.

$$\frac{1}{L} d(x,y) \le |\varphi(x) - \varphi(y)| \le L d(x,y)$$

for every $x, y \in B$.

3. Prove Lemma 4.34: Let $\gamma : [a,b] \to M$ be admissible and V a continuous piecewise smooth vector field along γ. Then there exists Γ, a variation of γ, such that V is the variation field of Γ. If V is proper, Γ can be taken to be proper as well.

4. Let $B(p,r) = \exp_p B(0,r)$ be a normal ball such that $\partial B(p,r)$ is a normal sphere ($= \exp_p \partial B(0,r)$). Prove that, for every $q \in M \setminus B(p,r)$, there exists a point $q' \in \partial B(p,r)$ such that $d(p,q) = r + d(q',q)$.

5. Generalize the first variation formula (Theorem 4.35) to the case of a variation that is not necessarily proper.