Department of Mathematics and Statistics
Riemannian geometry
Exercise 4
14.2.2013

1. Let M be a Riemannian manifold and let $U \subset M$ be an open set. The divergence of a vector field $X \in \mathcal{T}(U)$, denoted by $\operatorname{div} X$, is the trace of the linear map $Y \mapsto \nabla_Y X$. Thus $\operatorname{div} X : U \to \mathbb{R}$,

$$(\operatorname{div} X)(p) = \operatorname{tr}(v \mapsto \nabla_v X), \quad v \in T_pM.$$

Suppose that (U, x), $x = (x^1, \dots, x^n)$, is a chart. Express $\operatorname{div} X$ in local coordinates. [Recall that the trace of an $n \times n$ matrix (a_{ij}) is the sum of the diagonal entries $\sum_{i=1}^n a_{ii}$.]

2. Prove the claims (a), (d), and (e) in Lemma 4.19. That is: Suppose (U, φ) is a normal chart at p. Show that
   (a) If $v = v^i e_i \in T_pM$, then the normal coordinates of $\gamma^v(t)$ are $(tv^1, \dots, tv^n)$ whenever $tv \in \mathcal{V}$.
   (d) If $\varepsilon > 0$ is so small that $\exp_p$ is diffeomorphic in $B(0, \varepsilon) \subset T_pM$, then the set $\{x \in U : r(x) < \varepsilon\}$ is the normal ball $\exp_p\big(B(0, \varepsilon)\big)$.
   (e) If $q \in U \setminus \{p\}$, then $\left(\frac{\partial}{\partial r}\right)_q$ is the velocity vector ($= \dot\gamma$) of the unit speed geodesic from p to q in U (=unique by (a)), and therefore $|\frac{\partial}{\partial r}| \equiv 1$.

3. Prove the claims (c) and (f) in Lemma 4.19. That is: Suppose (U, φ) is a normal chart at p. Show that
   (c) The components of the Riemannian metric (with respect to the normal chart) at p are $g_{ij}(p) = \delta_{ij}$.
   (f) $\partial_k g_{ij}(p) = 0$ and $\Gamma^k_{ij}(p) = 0$.

4. Suppose that (M, g) and $(\tilde M, \tilde g)$ are Riemannian manifolds, and $\varphi\colon M \to \tilde M$ is an isometry. Let $\tilde\nabla$ be the Riemannian connection of $(\tilde M, \tilde g)$. Show that the mapping

$$\varphi^*\tilde\nabla \colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M),$$

$$\big(\varphi^*\tilde\nabla\big)_X Y := \varphi^*\tilde\nabla(X, Y) = \varphi_*^{-1}\big(\tilde\nabla_{\varphi_* X}(\varphi_* Y)\big),$$

is the Riemannian connection of (M, g).

5. Suppose that (M, g) and $(\tilde M, \tilde g)$ are Riemannian manifolds, and $\varphi\colon M \to \tilde M$ is an isometry.
   (a) Let $\gamma : I \to M$ be a smooth path, $\alpha = \varphi \circ \gamma$, and $\tilde D_t : \mathcal{T}(\alpha) \to \mathcal{T}(\alpha)$ the covariant derivative along α. Show that the mapping $\varphi^*\tilde D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$,

$$(\varphi^*\tilde D_t)V = \varphi_*^{-1}\big(\tilde D_t(\varphi_* V)\big),$$

is the covariant derivative along γ. [Here $\varphi_* V$, $(\varphi_* V)_t = \varphi_* V_t$, is a vector field along α.]

(b) Prove that φ maps geodesics to geodesics. That is, if γ is a geodesic on M such that $\gamma(0) = p$ and $\dot{\gamma}_0 = v$, then $\alpha = \varphi \circ \gamma$ is a geodesic on $\tilde{M}$ such that $\alpha(0) = \varphi(p)$ and $\dot{\alpha}_0 = \varphi_* v$.