Department of Mathematics and Statistics
Riemannian geometry
Exercise 3
7.2.2013

1. Let $\gamma\colon I \to M$ be a C^∞-path. For $t_0, t \in I$, define a mapping (linear isomorphism) $P_{t_0,t}\colon T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ by $P_{t_0,t}v = V(t)$, where $V \in \mathcal{T}(\gamma)$ is the parallel transport of $v \in T_{\gamma(t_0)}M$ along γ. Prove that
$$D_t W(t_0) = \lim_{t\to t_0} \frac{P_{t_0,t}^{-1}W(t) - W(t_0)}{t - t_0}$$
for $W \in \mathcal{T}(\gamma)$. [Hint: Use a parallel frame along γ.]

2. Let M be a Riemannian manifold, $\langle\cdot,\cdot\rangle$ the Riemannian metric, and ∇ the Riemannian connection of M. The Hessian of a real-valued function $u \in C^\infty(M)$ is a 2-covariant tensor field $\operatorname{Hess} f \in \mathcal{T}^2(M)$ defined by
$$\operatorname{Hess} f(X,Y) = \langle \nabla_X(\nabla f), Y\rangle, \quad X, Y \in \mathcal{T}(M).$$
Prove that $\operatorname{Hess} f$ is symmetric and
$$\operatorname{Hess} f(X,Y) = X(Yf) - (\nabla_X Y)f.$$

3. (a) Prove that the mapping $L\colon \mathcal{T}(M)\times\mathcal{T}(M) \to \mathcal{T}(M)$,
$$L(X,Y) = L_X Y$$
(= the Lie derivative of Y with respect to X) is not a connection.
   (b) Prove that there exist smooth vector fields $V \in \mathcal{T}(\mathbb{R}^2)$ and $W \in \mathcal{T}(\mathbb{R}^2)$ such that $V = W = \frac{\partial}{\partial x}$ along the x-axis, but the Lie derivatives $L_V\left(\frac{\partial}{\partial y}\right)$ and $L_W\left(\frac{\partial}{\partial y}\right)$ are not equal on the x-axis. (Conclusion?)

4. Let $Y \in \mathcal{T}(M)$ be a vector field on a Riemannian manifold M such that $|Y|$ is constant. Prove that $\nabla_X Y$ is always perpendicular to Y (i.e. $\langle \nabla_X Y, Y\rangle = 0$) for all $X \in \mathcal{T}(M)$.

5. Let $A_{ij}\colon \mathbb{R}^m \to \mathbb{R}$, $i,j = 1,\dots,n$, be smooth mappings and denote $A = (A_{ij})$. Prove that in the open set $\{x \in \mathbb{R}^m\colon \det A > 0\}$ we have
$$\frac{\partial}{\partial x^k}\log\det A = \operatorname{tr}\left(\frac{\partial A}{\partial x^k}A^{-1}\right)$$
for all $k = 1,\dots,m$. [Recall that the trace of an $n\times n$ matrix (a_{ij}) is the sum of the diagonal entries $\sum_{i=1}^n a_{ii}$.]

Note: Recall, for instance from the course "Introduction to differential geometry“, that $L_X Y = [X,Y]$.