Department of Mathematics and Statistics
Riemannian geometry
Exercise 12
2.5.2013

The course can be passed by an exam (e.g. May 13 (12:00-16:00) or May 16 (12:00-16:00)).

1. Let M be a Riemannian manifold. Suppose that $f\colon M \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ are smooth. Show that
$$\operatorname{Hess}(h \circ f) = (h'' \circ f) df \otimes df + (h' \circ f) \operatorname{Hess} f.$$

2. Suppose that $\kappa < 0$ is a constant. Prove that there exists on $\mathbb{R}^n$, $n \geq 2$, a Riemannian metric g such that $(\mathbb{R}^n, g)$ is complete and of constant sectional curvature κ.

3. Recall the exercise 10/5. Prove: If (a) or (b) is satisfied, then there exists $t' \in (0, t_0]$ such that $\gamma^v(t')$ is the cut point of p along γ^v.

4. Let M be a complete Riemannian manifold, $p \in M$, and $v \in T_pM$, with $|v| = 1$. Prove:
   (a) If q is a cut point of p along γ^v, then p is the cut point of q along the geodesic $-\gamma\colon [0, d(v)] \to M$, $-\gamma(t) = \gamma^v(d(v) - t)$. In particular, $q \in C(p)$ if and only if $p \in C(q)$.
   (b) If $q \in M \setminus C(p)$, there exists a unique minimizing geodesic from p to q.

5. Suppose that M is complete and that there exists $p \in M$ such that p has a cut point along every geodesic starting at p. Prove that M is compact.