Department of Mathematics and Statistics
Riemannian geometry
Exercise 11
25.4.2013

1. Let M and $\tilde{M}$ be n-dimensional Riemannian manifolds of constant sectional curvature κ. Prove that for any $p \in M$ and $\tilde{p} \in \tilde{M}$ there exists $\delta > 0$ such that the geodesic balls $B(p, \delta) \subset M$ and $B(\tilde{p}, \delta) \subset \tilde{M}$ are isometric.

2. Let (U, x) be a chart on a Riemannian manifold and let $f \in C^\infty(U)$ be a smooth real valued function. Compute $\operatorname{Hess} f$ in local coordinates and verify that $\operatorname{Hess} f$ is symmetric.

3. Let f be a smooth real valued function on a Riemannian manifold. Prove that
$$\Delta f = \operatorname{div}(\nabla f) = \operatorname{tr} \operatorname{Hess} f$$
with respect to the Riemannian metric.

4. Prove that, for a complete Riemannian manifold, $\operatorname{inj}(p) = d(p, C(p))$ provided $C(p) \neq \emptyset$, where $C(p)$ is as in Definition 8.15.

5. Prove that $p \mapsto \operatorname{inj}(p)$ is a continuous positive function on any Riemannian manifold.