Department of Mathematics and Statistics
Riemannian geometry
Exercise 10
18.4.2013

1. Let M be a connected complete Riemannian manifold such that $K(\sigma) \leq 0$ for every 2-dimensional subspace $\sigma \subset T_pM$ and for every $p \in M$. Prove by using Rauch's theorem that $\exp_p : T_pM \to M$ is a local diffeomorphism for every $p \in M$.

2. Suppose that M has constant sectional curvature κ. Let $p \in M$ and let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM, (U, φ) the corresponding normal chart at p such that U is a normal ball, and g_{ij} the corresponding component functions of the Riemannian metric. Prove that
$$g_{ij}(\exp_p v) = \frac{v^i v^j}{|v|^2} + \frac{\mathbf{S}_\kappa^2(|v|)}{|v|^2}\left(\delta_{ij} - \frac{v^i v^j}{|v|^2}\right),$$
for $\exp_p v \in U$, $v = v^i e_i \neq 0$. Here
$$\mathbf{S}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, t), & \kappa > 0, \\ t, & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, t), & \kappa < 0. \end{cases}$$

3. Prove Lemma 8.22: Let M be an oriented Riemannian manifold, ω_M its Riemannian volume form, and $V \in \mathcal{T}(M)$. Then the divergence of V,
$$\operatorname{div} V = \operatorname{tr}(X \mapsto \nabla_X V)$$
satisfies
$$L_V \omega_M = (\operatorname{div} V)\omega_M.$$
[Recall "Cartan's magic formula": $L_V\alpha = d i_V \alpha + i_V d\alpha$.]

4. Let M be a Riemannian n-manifold, $p \in M$, and $r(x) = d(x, p)$. Prove that
$$\Delta r(x) = \frac{n-1}{r(x)} + O\big(r(x)\big)$$
as $x \to p$.

Let M be a complete Riemannian manifold, $p \in M$, and $v \in T_pM$, with $|v| = 1$. Recall that the distance to the cut point of p along γ^v is defined as
$$d(v) = \sup\{t > 0 \colon tv \in \mathcal{E}_p, d\big(p, \gamma^v(t)\big) = t\}.$$
If $d(v) < \infty$, we say that $\gamma^v\big(d(v)\big)$ is the cut point of p along γ^v.

5. Suppose that $\gamma^v(t_0)$ is the cut point of $p = \gamma^v(0)$ along γ^v. Prove that at least one of the following conditions holds

(a) $\gamma^v(t_0)$ is the first conjugate point of p along γ^v, or
(b) there exists a unit speed geodesic $\sigma \neq \gamma^v|[0, t_0]$ from p to $\gamma^v(t_0)$ such that $\ell(\sigma) = t_0 = \ell\big(\gamma^v|[0, t_0]\big)$.