Department of Mathematics and Statistics
Riemannian geometry
Exercise 1
24.1.2013

1. The standard Riemannian metric of the Euclidean space $\mathbb{R}^n$ is
$$g = \delta_{ij} dx^i dx^j.$$
Write g in the case of the plane $\mathbb{R}^2$ by using the polar coordinates $(r, \vartheta),\ x^1 = r\cos\vartheta,\ x^2 = r\sin\vartheta$.

2. A surface of revolution is obtained as follows: Consider a smooth simple (i.e. injective) curve $\gamma = (r, z)\colon I \to \mathbb{R}^2,\ \gamma(t) = \big(r(t), z(t)\big)$, where $I \subset \mathbb{R}$ is an interval. By rotating this curve around z-axis, we get a surface whose cylindrical coordinate representation is
$$(t, \vartheta) \mapsto f(t, \vartheta) = \big(r(t)\cos\vartheta, r(t)\sin\vartheta, z(t)\big).$$
We obtain a natural frame $\partial_t, \partial_\vartheta$ with coframe $dt, d\vartheta$. Write the induced Riemannian metric $dx^2 + dy^2 + dz^2$ from $\mathbb{R}^3$ in this frame.

3. Let (M, g) be a Riemannian n-manifold. Prove that for every $p \in M$ there exist a neighborhood $U \ni p$ and smooth vector fields $E_1, \dots, E_n \in \mathcal{T}(U)$ that form a local orthonormal frame in U, i.e. the vectors $(E_1)_q, \dots, (E_n)_q$ form an orthonormal basis for T_qM for every $q \in U$. [Hint: Use the Gram-Schmidt algorithm.]

Definitions for problems 4 and 5:

(a) Let $(M, g),\ g = \langle\cdot,\cdot\rangle$, be a Riemannian manifold, $U \subset M$ open, and $f \in C^\infty(U)$ a real-valued function. The *gradient* of f is a vector field $\nabla f \in \mathcal{T}(U)$ such that
$$\langle \nabla f, X\rangle = Xf$$
for every $X \in \mathcal{T}(U)$.

(b) We say that a Riemannian metric $\hat{g} = \langle\langle\cdot,\cdot\rangle\rangle$ is obtained from the Riemannian metric $g = \langle\cdot,\cdot\rangle$ by a conformal change if
$$\langle\langle\cdot,\cdot\rangle\rangle_p = \varphi(p)\langle\cdot,\cdot\rangle_p \tag{1}$$
for every $p \in M$, where $\varphi\colon M \to]0,\infty[$ is a (positive) C^∞-function.

4. Let $(M, g),\ g = \langle\cdot,\cdot\rangle$, be a Riemannian manifold and let $(U, x),\ x = (x^1, \dots, x^n)$, be a chart. Express the gradient, ∇f, of a C^∞-function $f\colon U \to \mathbb{R}$ in local coordinates, i.e. by using the coordinate frame $\partial_1, \dots, \partial_n$.

5. Let $(M^n, g),\ g = \langle\cdot,\cdot\rangle$, be a Riemannian n-manifold. Suppose that $\hat{g} = \langle\langle\cdot,\cdot\rangle\rangle$ is obtained from g by a conformal change (1). Let $f\colon M^n \to \mathbb{R}$ be a C^∞-function, ∇f its gradient with respect to

the Riemannian metric $\langle\cdot,\cdot\rangle$, and $|\nabla f| = \langle \nabla f, \nabla f\rangle^{1/2}$. Furthermore, let $\hat{\nabla} f$ be the gradient of f with respect to the Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ and $\|\hat{\nabla} f\| = \langle\langle \hat{\nabla} f, \hat{\nabla} f\rangle\rangle^{1/2}$.

(a) Express $\hat{\nabla} f$ and $\|\hat{\nabla} f\|$ in terms of the function φ in formula (1), ∇f, and $|\nabla f|$.

(b) Let (U, x) be a chart. Compare the integrals

$$\int_U |\nabla f|^n \, d\mu \tag{2}$$

and

$$\int_U \|\hat{\nabla} f\|^n \, d\hat{\mu}. \tag{3}$$

What do you notice? Above in (2) we integrate with respect to the Riemannian metric g and in (3) with respect to the Riemannian metric $\hat{g}$. In other words,

$$\int_U |\nabla f|^n \, d\mu = \int_{xU} \left(|\nabla f|^n \sqrt{\det(g_{ij})} \right) \circ x^{-1} \, dm$$

and

$$\int_U \|\hat{\nabla} f\|^n \, d\hat{\mu} = \int_{xU} \left(\|\hat{\nabla} f\|^n \sqrt{\det(\hat{g}_{ij})} \right) \circ x^{-1} \, dm.$$