6.8. Remarks a) It is possible to show (see [AIM, p.97]) that for $f \in L^p(\mathbb{C})$ the limit defining the principal value integral,

$$\text{p.v.}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta-z)^2}\, dm := \lim_{\varepsilon \to 0} \int_{|\zeta - z|>\varepsilon} \frac{f(\zeta)}{(\zeta-z)^2} \quad \text{exists for a.e. } z \in \mathbb{C}$$

This makes Sf a well defined object for L^p-fcn's, too.

b) For $g \in C_0^\infty(\mathbb{C})$ we have

$$(6.6.) \qquad S(g_{\bar z}) = g_z \quad , \text{i.e. } "S \circ \bar\partial = \partial"$$

Indeed, $S(g_{\bar z}) := \partial \underbrace{\mathcal{C}(g_{\bar z})}_{= g \text{ by } (6.1)} = g_z$.

To extend (6.6) to Sobolev functions we need

6.9. Theorem $\|Sf\|_{L^2(\mathbb{C})} = \|f\|_{L^2(\mathbb{C})} \quad \forall f \in L^2(\mathbb{C})$.

Proof: If $g \in C_0^\infty(\mathbb{C})$, then

$$\int_{\mathbb{C}} |g_z|^2 = \int_{\mathbb{C}} g_z \bar g_{\bar z} \underset{\substack{\uparrow \\ \text{int. by parts}}}{=} \int_{\mathbb{C}} g_{\bar z} \bar g_z = \int_{\mathbb{C}} |g_{\bar z}|^2 .$$

(Thus have shown: $\int_{\mathbb{C}} J(z,g) = 0$ for $g \in C_0^\infty(\mathbb{C})$!)

So for $f = g_{\bar z}$, $g \in C_0^\infty(\mathbb{C})$, we have the claim. But such f's are dense in L^2: If $\underbrace{\quad}_{(f \in L^2(\mathbb{C}) \text{ and}}$ $f \perp \{g_{\bar z} : g \in C_0^\infty(\mathbb{C})\}$ then

$$0 = \langle \varphi_{\bar z}, f \rangle_{L^2} = \int_{\mathbb{C}} \varphi_{\bar z}\, f \, dm \qquad \forall\, \varphi \in C_0^\infty(\mathbb{C}), \text{ i.e.}$$

$\bar\partial f = 0$ in the weak sense. By the remark (p. 50) after Weyl's lemma it follows f analytic in $\mathbb{C}$. But as $f \in L^2(\mathbb{C}) \Rightarrow f$ bounded $\Rightarrow f \equiv \text{const} = 0$.

By the density of $\{g_{\bar z} : g \in C_0^\infty(\mathbb{C})\}$ in $L^2(\mathbb{C})$, Beurling op. extends to an isometry of all of $L^2(\mathbb{C})$,

$$\int_{\mathbb{C}} |Sf|^2 \, dm = \int_{\mathbb{C}} |f|^2 \, dm \qquad \forall\, f \in L^2(\mathbb{C})$$

$\square$

<u>Note</u>: One can use the isometry property to define Sf for all $f \in L^2$, by the density as above.

Much deeper lie the L^p-properties of S. We have

<u>6.10. Theorem</u> The Beurling transform extends to a continuous operator on $L^p(\mathbb{C})$, $1 < p < \infty$,

$$\| Sf \|_{L^p(\mathbb{C})} \leq \|S\|_p \cdot \|f\|_{L^p(\mathbb{C})}, \qquad f \in L^p(\mathbb{C}),$$

for some constant $\|S\|_p$ depending only on p, with $p \mapsto \|S\|_p$ continuous.
(for proof see e.g. [AIM])

<u>Remark</u>: The famous Iwaniec conjecture asserts that (take) can

$$\|S\|_p = \max\left\{p-1, \frac{1}{p-1}\right\}, \qquad 1 < p < \infty.$$

Remark. The L^p-properties of the Cauchy transform are little more "subtle". For instance, by theory of Riesz potentials

(6.7) $$\| \mathcal{C} f \|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \le c_p \| f \|_{L^p(\mathbb{C})}, \quad 1 < p < 2$$

For $f \in L^p(\mathbb{C})$, $2 \le p < \infty$, the integral $\int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} \, dm(\zeta)$ may not converge at ∞ (as $\frac{1}{\zeta} \notin L^q(|\zeta| > 1)$, $\frac{1}{p} + \frac{1}{q} = 1$). Thus we set now

$$\mathcal{C} f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \underbrace{\left[\frac{1}{\zeta - z} - \frac{1}{\zeta} \chi_{\mathbb{C} \setminus \mathbb{D}}(\zeta) \right]}_{\overset{|\zeta|>1}{=} \frac{z}{\zeta(\zeta - z)} \in L^q(|\zeta| > 1)} dm(\zeta)$$

This differs from "old" $\mathcal{C} f(z)$ only by a constant; thus still $\bar{\partial} \, \mathcal{C}(f) = f$!

6.11. Corollary. If $f \in L^p(\mathbb{C})$ with $f(z) = 0$ for $|z| > R$, then

$$\| \mathcal{C} f \|_{L^p(B(0, 2R))} \le C(p) R \| f \|_p, \quad R > 1, \; 1 < p < \infty.$$

In particular, if $L^p_0(\mathbb{C}) = \{ f \in L^p(\mathbb{C}), \operatorname{supp}(f) \text{ bounded} \}$, then

$$\mathcal{C} : L^p_0(\mathbb{C}) \longrightarrow \mathring{W}^{1,p}(\mathbb{C}) = \{ f \in W^{1,p}_{loc}(\mathbb{C}) : \nabla f \in L^p(\mathbb{C}) \}$$

Proof: If $K_R(\zeta) = \frac{-1}{\pi} \frac{1}{\zeta} \chi_{B(0,3R)}(\zeta)$, then for $|z| < 2R$,

$$K_R * f(\zeta) = -\frac{1}{\pi}\int_{|\zeta - z|<3R} \frac{1}{\zeta - z} f(\zeta) = \mathcal{C}f(z) \quad . \text{ Thus}$$

$$\| \mathcal{C}f \|_{L^p(B(0,2R))} \leq \| K_R * f \|_{L^p(\mathbb{C})} \leq \underbrace{\| K_R \|_{L^1}}_{= CR} \| f \|_{L^p(\mathbb{C})} .$$

Further, as $\| \bar\partial \mathcal{C}f \|_{L^p} = \| f \|_{L^p}$ and $\| \partial \mathcal{C}f \|_{L^p} = \| Sf \|_{L^p} \leq c \| f \|_{L^p}$ we have $\| \nabla \mathcal{C}f \|_{L^p} \leq c \| f \|_{L^p} < \infty$. $\square$

<u>6.12. Corollary</u> $\quad S(g_{\bar z}) = g_z \quad \forall\, g \in \mathring{W}^{1,p}(\mathbb{C})$.

<u>Pf.</u> Follows from (6.6) & Theorem 6.10 . $\square$

Corollary 6.12 makes it possible to calculate $S(h)$ for some concrete functions:

<u>Example</u> Determine $\quad S(\chi_{\mathbb{D}}) = ?$

<u>Sol'n</u>: If $g(z) = \begin{cases} \bar z , & |z|<1 \\ \frac{1}{z} , & |z|>1 \end{cases}$ then $g \in \mathring{W}^{1,p}(\mathbb{C})$ for $1<p<\infty$ and $g_{\bar z} = \chi_{\mathbb{D}}$. Thus $S(\chi_{\mathbb{D}}) = g_z = \frac{-1}{z^2} \chi_{\mathbb{C}\setminus\mathbb{D}}$

6.13. Lemma. If

- $|\mu(z)| \le k\,\chi_{B(0,R)}(z)$ for some $R<\infty$, and
- $\varphi \in L^2_0(\mathbb{C})$,

then the equation

(6.8) $\qquad \bar\partial\sigma = \mu\,\partial\sigma + \varphi$

has a unique solution $\sigma \in \mathring{W}^{1,2}(\mathbb{C})$ with $\sigma(z) = O(\frac{1}{z})$ at ∞

Proof: By Cor. 6.12, (6.8) $\Leftrightarrow$ $\sigma_{\bar z} = \mu S(\sigma_{\bar z}) + \varphi$ $\Leftrightarrow$ $(I-\mu S)(\sigma_{\bar z}) = \varphi$.

As $\|\mu\|_\infty \|S\|_{L^2\to L^2} \le k\cdot 1 = k < 1$, the usual Neumann series argument shows $(I-\mu S)^{-1} : L^2 \to L^2$.

Thus get

$$\sigma_{\bar z} = (I-\mu S)^{-1}\varphi = \varphi + \mu S(\varphi) + \mu S \mu S(\varphi) + \cdots$$

where the converges in $L^2(\mathbb{C})$-norm; in fact sum $\in L^2_0(\mathbb{C})$

Define now

$$\sigma(z) := \mathcal{C}\left((I-\mu S)^{-1}\varphi\right)$$

By Cor. 6.11 $\sigma \in \mathring{W}^{1,2}(\mathbb{C})$ and $\sigma(z) = \mathcal{C}(\sigma_{\bar z}) = O(1/z)$. (why?)

Further, $\sigma_{\bar z} = (I-\mu S)^{-1}\varphi \Rightarrow \sigma_{\bar z} - \mu S(\sigma_{\bar z}) = \sigma_{\bar z} - \mu\sigma_z = \varphi$.

Uniqueness If σ_1 another sol'n,

$$(\sigma-\sigma_1)_{\bar z} = \mu(\sigma-\sigma_1)_z \Rightarrow (I-\mu S)\Big(\underbrace{(\sigma-\sigma_1)_{\bar z}}_{\in L^2}\Big) = 0 \Rightarrow$$

$(\sigma-\sigma_1)_{\bar z} = 0 \Rightarrow \sigma \equiv \sigma_1$ (c.f. p. 58). $\square$

Recall the notation: $\mathring{W}^{1,2}(\mathbb{C}) = \{ g \in W^{1,2}_{loc}(\mathbb{C}) : \nabla g \in L^2(\mathbb{C}) \}$.

6.14. Theorem. If $|\mu(z)| \le k \chi_{B(0,R)}(z)$, $z \in \mathbb{C}$, then there is a unique $f \in W^{1,2}_{loc}(\mathbb{C})$ with

(6.9) $\quad \bar\partial f = \mu(z) \partial f$ a.e. $z \in \mathbb{C}$ & $f(z) = z + O(1/z)$, $|z| \to \infty$.

Proof: Use Lemma 6.13 with $\varphi \equiv \mu \in L^2_0(\mathbb{C})$ and define

$$f(z) := z + \sigma(z) \underset{\text{by proof of 6.13}}{=} z + \mathcal{C}\left((I - \mu S)^{-1} \mu \right)$$

$$= z + \mathcal{C}\left(\mu + \mu S \mu + \mu S \mu S \mu + \dots \right)$$

As $(I - \mu S)^{-1} \mu \in L^2_0(\mathbb{C})$, $f(z) = z + O(1/z)$ as $|z| \to \infty$.

Also $f_{\bar z} = \sigma_{\bar z} \underset{\text{Le 6.13}}{=} \mu \sigma_z + \mu = \mu(1 + \sigma_z) = \mu f_z$.

Uniqueness: If f_1 another solution $\Rightarrow \underset{= O(1/z^2) \text{ (at } \infty)}{\nabla(f - f_1)} \in L^2(\mathbb{C})$

and $(I - \mu S)\left((f - f_1)_{\bar z} \right) = (f - f_1)_{\bar z} - \mu (f - f_1)_z = 0$.

As $I - \mu S$ invertible $\Rightarrow (f - f_1)_{\bar z} = 0$ in $L^2 \Rightarrow f - f_1$ analytic (c.f. p. 51)

As $\left. \begin{array}{l} f - f_1 = O(1/z) \\ f - f_1 \text{ analytic} \end{array} \right\} \Rightarrow f \equiv f_1$. $\square$

6.15. Definition. When $\mathrm{supp}(\mu)$ compact, solutions to (6.9) with $f(z) = z + O(1/z)$ are called principal solutions.

Question: Are principal solutions homeomorphic?

For proving the homeomorphicity we need first:

6.16. Lemma. Let $\mu, \varphi \in C_0^\infty(\mathbb{C})$ and $\sigma_{\bar z} = \mu\sigma_z + \varphi$ as in Lemma 6.13; $|\mu| \le k < 1$ and $\sigma \in \mathring{W}^{1,2}(\mathbb{C})$.

Then $\sigma \in C^\infty(\mathbb{C})$.

Proof: Note first that $\bar\partial S(\varphi) = \bar\partial \partial \mathcal{C}(\varphi) = S(\bar\partial\varphi)$ for any $\varphi \in C_0^\infty(\mathbb{C})$; similarly $\partial S(\varphi) = S(\partial\varphi)$.

On the other hand, proof of Le. 6.13 $\Rightarrow$

$$\sigma_{\bar z} = \varphi + \mu S(\varphi) + \mu S \mu S(\varphi) + \dots = w_0 + w_1 + w_2 + \dots$$

If $D = \alpha\partial + \beta\bar\partial$ (α, β constant) then

$$D w_n = D\mu\, S(\mu S \mu \dots \mu S\varphi) + \mu S(D\mu\, S(\mu S\mu \dots \mu S\varphi)) + \dots + \mu S(\mu S(\mu \dots \mu S D\varphi))$$

Thus

$$\| D w_n \|_{L^2(\mathbb{C})} \le (n+1) \underbrace{\| S \|_{L^2}^n}_{=1} k^{n-1} \left(\|\varphi\|_{L^2} \|D\mu\|_{L^\infty} + \|D\varphi\|_{L^2} \right)$$

and get

$$h := \sum_{n=0}^\infty D w_n \in L^2(\mathbb{C})$$ where sum abs. convergent in the L^2-norm. But then

$$\int_{\mathbb{C}} D\varphi\, \sigma_{\bar z} = - \int_{\mathbb{C}} \varphi\, h \quad \forall \varphi \in C_0^\infty(\mathbb{C})$$

is $\sigma_{\bar z}$ has L^2-derivatives; $\sigma_{\bar z} \in W^{1,2}(\mathbb{C})$. Can continue this inductively $\Rightarrow$ $\sigma_{\bar z} \in \bigcap_{k=0}^\infty W^{k,2}(\mathbb{C}) \subset C^\infty(\mathbb{C})$. Here the last inclusion uses Fourier-analysis:

$$\phi \in \bigcap_{k=0}^{\infty} W^{k,2}(\mathbb{C}) \iff \int_{\mathbb{C}} (1+|\xi|^2)^k \, |\hat{\phi}(\zeta)|^2 \, dm < \infty \quad \forall\, k \in \mathbb{N},$$

with $f \in C^1(\mathbb{R}^2)$ whenever $|\zeta|\,|\hat{f}(\zeta)| \in L^1(\mathbb{R}^2)$

(or refer to Sobolev embedding).

Finally, $\sigma(z) = \mathcal{C}(\sigma_{\bar z}) \in C^\infty(\mathbb{C})$ [Remark 6.6]

$\square$

Now we are ready for:

<u>6.17. Theorem</u> (Measurable Riemann Mapping Th. for $\mu \in C_0^\infty$)

Suppose $\mu \in C_0^\infty(\mathbb{C})$ with $|\mu(z)| \le k < 1$, $z \in \mathbb{C}$.

Let $f \in W^{1,2}_{loc}(\mathbb{C})$ be the principal solution to

$$(6.10) \qquad \bar\partial f = \mu\, \partial f, \quad \text{a.e. } z \in \mathbb{C}.$$

Then

1°) f is a C^∞-diffeomorphism of $\mathbb{C}$ $\Rightarrow$ K-q conformal $\left(K = \frac{1+k}{1-k}\right)$

2°) $J(z, f) > 0 \quad \forall\, z \in \mathbb{C}$.

<u>Proof:</u> $f(z) = z + \mathcal{C}\left((I-\mu S)^{-1}\mu\right) \in C^\infty(\mathbb{C})$ by previous lemma. The main point is to prove 2°). For this

solve the auxiliary equation

$$\sigma_{\bar z} = \mu \sigma_z + \mu_z$$

for $\sigma \in C^\infty(\mathbb{C}) \cap \mathring{W}^{1,2}(\mathbb{C})$, as in Lemma 6.16. Define now

$$F(z) = z + \mathcal{C}(\mu e^\sigma) \tag{6.11}$$

As $\mu e^\sigma \in C_0^\infty(\mathbb{C})$ we have $F \in C^\infty(\mathbb{C})$ with $F(z) = z + O(1/z)$ as $|z| \to \infty$. We claim that F solves (6.10); then F will a principal solution to (6.10), and by uniqueness of such solutions it will follow $f(z) \equiv F(z)$.

For this purpose note first that

$$(\mu e^\sigma)_z = (\mu_z + \mu \sigma_z) e^\sigma = \sigma_{\bar z} e^\sigma = (e^\sigma)_{\bar z}$$

Moreover, then

why?!

$$e^{\sigma(z)} - 1 = \mathcal{C}((e^\sigma)_{\bar z}) = \mathcal{C}((\mu e^\sigma)_z)$$

$$= \partial \mathcal{C}(\mu e^\sigma) = S(\mu e^\sigma)$$

i.e. we have shown

$$\boxed{e^{\sigma(z)} = 1 + S(\mu e^\sigma)}$$

With this info we can calculate from (6.11) that

$$\bar\partial F = \mu e^\sigma \quad ; \quad \partial F = 1 + S(\mu e^\sigma) = e^\sigma$$

Thus $\bar\partial F = \mu \partial F$ and $f \equiv F$! But then

$$J(z, f) = |\partial F|^2 - |\bar\partial F|^2 = |e^{2\sigma}|(1 - |\mu|^2) \geq |e^{2\sigma}|(1-k^2) > 0 \quad \forall z \in \mathbb{C}$$

To complete 1°), by 2°) and inverse function theorem, f is a local homeomorphism in $\mathbb{C}$. As $f(z) = z + O(1/z)$, f is also a homeo in $\{|z| > R\}$ for some large R; for this consider $h(z) := 1/f(1/z) = \frac{z}{1+O(z)}$ for $|z| < \varepsilon$; as $h'(0) = 1$, h is homeo in a nbhood of origin $\Rightarrow$ f in a nbhood of ∞.

Thus f a local homeo of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the Riemann sphere. In particular f is an open mapping; thus $f\hat{\mathbb{C}}$ open and closed in $\hat{\mathbb{C}}$, hence $f\hat{\mathbb{C}} = \hat{\mathbb{C}}$. To prove injectivity one may use monodromy thm. and the fact that $\hat{\mathbb{C}}$ is simply connected. But there is also an elementary argument: Let

$$A(n) = \{ w \in \hat{\mathbb{C}} : \# f^{-1}(w) = n \} .$$

Then $A(n)$ is open and closed in $\hat{\mathbb{C}}$, thus either $A(n) = \hat{\mathbb{C}}$ or $A(n) = \emptyset$. As $f^{-1}(\infty) = \{\infty\}$ we have $A(1) = \hat{\mathbb{C}}$ and $A(n) = \emptyset$ for $n \neq 1$; in particular every $w \in \hat{\mathbb{C}}$ has only one preimage $\Rightarrow$ f is injective $\Rightarrow$ $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ homeo!

Last, as f is a homeomorphic (and smooth) solution to Beltrami eqn, f is K-quasiconformal (see p. 43)

$\square$

VI.4. Meas. Riemann Map. Theorem for general μ

For a general $\mu \in L^\infty(\mathbb{C})$, $|\mu| \le k < 1$, we need the so called "good approximation lemma":

6.18. Lemma. Suppose $\{\mu_n\}_{n\ge 1} \subset L^\infty(\mathbb{C})$ are such that $\|\mu_n\|_\infty \le k < 1$ $\forall n \ge 1$, and the limit

$$\mu(z) := \lim_{n\to\infty} \mu_n(z)$$

exists for a.e. $z \in \mathbb{C}$. Suppose further that $f_n : \mathbb{C} \to \mathbb{C}$ are $W^{1,2}_{loc}(\mathbb{C})$ - homeos solving

$$\bar\partial f_n = \mu_n(z)\, \partial f_n \quad \text{a.e. } z \in \mathbb{C} \ \ \& \ \ f_n(0) = 0,\ f_n(1) = 1.$$

(in part: f_n unique!). Then the limit

$$f(z) := \lim_{n\to\infty} f_n(z)$$

exists $\forall z \in \mathbb{C}$, the convergence is uniform on compact subsets of $\mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$ homeo, $f \in W^{1,2}_{loc}(\mathbb{C})$ and

$$\bar\partial f = \mu(z)\, \partial f \quad \text{a.e. } z \in \mathbb{C}. \tag{6.12}$$

Proof: By previous chapters, f_n are η_K-quasisymm., $K = \frac{1+k}{1-k} < \infty$. Also Exercise set 1 $\Rightarrow$ $\{f_n\}$ equicontinuous, pointwise bdd and there is a subsequence $\{f_{n_k}\}$ such that

$f_{\mu_k}(z) \to f(z)$ unif. on compacts of $\mathbb{C}$, where $f(z)$ is a η_K-quasisymmetric map of $\mathbb{C}$, with $f(0)=0$ and $f(1)=1$. It suffices to show that f solves (6.12), since the normalized solution is unique, and thus all converging subsequences have the same limit!

Let us first show that derivatives of f_{μ_k} converge weakly in $L^2_{loc}(\mathbb{C})$. Indeed, for any $R<\infty$,

$$(6.13)\quad \int_{B(0,R)} |Df_n|^2 \leq K \int_{B(0,R)} J(z,f_n) = K\,|f B_n(0,R)|$$

$$\leq \pi K \eta(R)^2 ;$$

thus for a further subsequence $\partial f_{\mu_k} \to g$ weakly in $L^2_{loc}(\mathbb{C})$, and similarly $\bar\partial f_{\mu_k} \to h \in L^2_{loc}(\mathbb{C})$.

We claim that $g = \partial f$, $h = \bar\partial f$. Namely, $\forall\, \varphi \in C_0^\infty(\mathbb{C})$

$$\int \partial\varphi\, f = \lim_k \int_{\mathbb{C}} \partial\varphi\, f_{\mu_k} = -\int_{\mathbb{C}} \varphi\, g \;\Rightarrow\; g = \partial f.$$ Same reasoning $\Rightarrow h = \bar\partial f$.

Further, if $\operatorname{supp}(\varphi) \subset B(0,R)$

$$(6.14)\quad \int_{\mathbb{C}} \varphi\left(\bar\partial f_{\mu_k} - \mu\, \partial f_{\mu_k}\right) = \int_{\mathbb{C}} \varphi\,(\mu_{n_k} - \mu)\, \partial f_{\mu_k}$$

Here L.H.S $\xrightarrow[(k\to\infty)]{} \int_{\mathbb{C}} \varphi\,(\bar\partial f - \mu\,\partial f)$, as the derivatives converge weakly in $\underline{L^2}(B(0,R))$. On the other hand

R.H.S. of (6.14) $\leq$

$$\| \mathcal{C}(\mu_n - \mu) \|_{L^2} \; \| D f_{n_k} \|_{L^2(B(0,R))} \overset{(6.13)}{\leq}$$

$$\sqrt{\pi K}\, \eta(R) \, \| \phi(\mu_n - \mu) \|_{L^2(\mathbb{C})} \longrightarrow 0$$

Dom. konvergence !

Thus have shown that any limit f satisfies (6.12). $\square$

<u>6.19. Remark</u> The normalization $f_n(0)=0$, $f_n(1)=1$ was used just to get rel. compactness of $\{f_n\}_{n\geq 1}$. If any other normalization gives compactness (and uniqueness), then the proof of Lemma 6.18 works as such in this situation.

<u>6.20. Theorem</u> (M.R.M.T.) If $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \leq k < 1$, then there exists a unique $W^{1,2}_{loc}(\mathbb{C})$-homeo $f: \mathbb{C} \to \mathbb{C}$ with

$$\bar\partial f = \mu \partial f \quad \& \quad f(0)=0, \; f(1)=1.$$

<u>Proof</u>: Choose $\mu_n \in C_0^\infty(\mathbb{C})$ with $\mu_n(z) \to \mu(z)$ a.e. z, $\|\mu_n\|_\infty \leq k < 1$. Solve then $\bar\partial f_n = \mu_n \partial f$ with $f_n : \mathbb{C} \to \mathbb{C}$ homeo (use Thm. 6.17), $f_n(0)=0$, $f_n(1)=1$

[If F_μ corresp. principal sol'n, let $f_\mu(z) = \dfrac{F_\mu(z)-F_\mu(0)}{F_\mu(1)-F_\mu(0)} = \alpha F_\mu + \beta$]. Use Lemma 6.18 to get existence and Corollary 6.4 or Remark 6.5 to get uniqueness. ∎

VII General solutions to Beltrami eq's

In this section we study and classify solutions $g \in W^{1,2}_{loc}(\Omega)$ (or even g with less regularity) to

$$g_{\bar z} = \mu\, g_z \quad , \quad |\mu| \le k < 1 ,$$

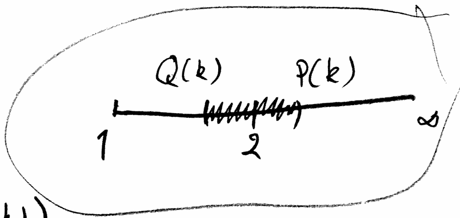
<u>without</u> any further assumptions (such as homeo etc.)

VII.1. L^p-estimates & Caccioppoli Inequalities

Start with further norm bounds for Cauchy & Beurling op.

A. Recall $S: L^p(\mathbb{C}) \to L^p(\mathbb{C})$ bounded with norm $\|S\|_p < \infty$ for $1<p<\infty$. Also $\|S\|_2 = 1$ and Riesz-Thorin thm $\Rightarrow$ $p \mapsto \|S\|_p$ continuous. Thus there exists exponents

$$(7.1) \qquad Q(k) < 2 < P(k)$$



such that $k\|S\|_p < 1$ whenever $p \in (Q(k), P(k))$.

We call $(Q(k), P(k))$ the <u>critical interval</u>,

Note: as $k \to 0$, $(Q(k), P(k)) \to (1, \infty)$

as $k \to 1$, $(Q(k), P(k)) \to \{2\}$.

If Iwaniec conjecture true $\Rightarrow$ $(Q(k), P(k)) = (1+k, 1+1/k)$.

B. • If $f \in L^p(\mathbb{C})$, $1 < p < 2$, $\Rightarrow$

(7.2) $\quad \mathcal{C}f \in L^r(\mathbb{C})$, $r = \frac{2p}{2-p} > 2$. (Sobolev embedding)

• If $f \in L^2(\mathbb{C}) \Rightarrow \mathcal{C}f \in VMO(\mathbb{C})$; in fact for some $a > 0$,

$$\int_B e^{a|\mathcal{C}f|^2} < \infty \quad \forall \text{ disk } B \subset \mathbb{C}. \quad \text{(Trudinger)}$$

• If $f \in L^p(\mathbb{C})$, $p > 2 \Rightarrow \mathcal{C}f \in Lip_\alpha$, $\alpha = 1 - 2/p$

(The proof of the last claim is part of Exercise set 3)

<u>7.1. Example</u> on using the critical interval: If $0 \le k < 1$ let $\mathcal{F}_k := \{f : \mathbb{C} \to \mathbb{C}$ principal sol'n, $\bar\partial f = \mu \partial f$, $|\mu| \le k \chi_{\mathbb{D}}\}$

We claim that $\mathcal{F}_k$ is compact in topo of unif. convergence on compacts of $\mathbb{C}$. Once this is shown, Thm 6.14, Thm 6.17 and Remark 6.19 show that <u>any</u> principal soln.

(for μ smooth or not) is a <u>homeomorphism</u> !

For compactness of $\mathcal{F}_k$, recall from proof of Thm. 6.14 that $f \in \mathcal{F}_k \Rightarrow$

$$\bar\partial f = (I - \mu S)^{-1}\mu \quad \& \quad f(z) = z + \mathcal{C}\big((I-\mu S)^{-1}\mu\big)$$

Here

$$\|(I-\mu S)^{-1}\mu\|_{L^p(\mathbb{C})} \le C(p)\,\|\mu\|_{L^p(\Delta)} \le \pi^{1/p} C(p),$$

$\forall\, p \in (Q(k), P(k))$. Choosing $2 < p < P(k)$, Exercise 3 $\Rightarrow$

$$|f(z) - f(w)| \le |z-w| + C_1(p)\,\|f_{\bar z}\|_{L^p(\mathbb{C})}\,|z-w|^{1-2/p} \le |z-w| + C_2\,|z-w|^{1-2/p},$$

for all $f \in \mathcal{F}_k$. Thus $\mathcal{F}_k$ is equicontinuous. But by Thm. 2.10.4 on p. 13A, $\mathcal{F}_k$ is pointwise bdd. Thus Ascoli-Arzela $\Rightarrow$ $\mathcal{F}_k$ compact!

<u>7.2. Theorem</u> (Caccioppoli inequalities). Suppose $0 \le k < 1$, and $f \in W^{1,q}_{loc}(\Omega)$, where $q \in (Q(k), P(k))$. Assume

$$|\bar\partial f(z)| \le k\,|\partial f(z)| \quad \text{a.e. } z \in \Omega. \tag{7.3}$$

Then, if $\eta \in C_0^\infty(\Omega)$, we have

$$\|\eta\, Df\|_{L^s(\mathbb{C})} \le C(s,k)\,\|f\, D\eta\|_{L^s(\mathbb{C})} \quad \forall\, s \in (Q(k), P(k)).$$

Thus $f \in W^{1,s}_{loc}(\Omega)$. In particular, $f \in C(\Omega)$!

Remark $W^{1,p}_{loc}(\Omega) \subset C(\Omega)$ when $p>2$ (Exercises 3)
but not when $p \le 2$; take e.g. $f(z) = \log|\log|z||$, $|z|<1$;
$|\nabla f| \simeq \frac{1}{|z||\log|z||} \in L^2_{loc}(\mathbb{D})$.

Proof of Thm 7.2: (7.3) $\Rightarrow$ $f_{\bar z} = \mu f_z$ a.e. $z \in \Omega$,
$|\mu| \le k < 1$. As $\eta \in C_0^\infty(\Omega)$,

$$F := \eta f \in W^{1,q}(\mathbb{C}) \cap L^t(\mathbb{C}),$$

for some $2 < t < P(k)$ [use (7.2) / Sobolev embedding]
Further,

$$F_{\bar z} = \mu F_z + \underbrace{(\eta_{\bar z} - \mu \eta_z) f}_{\equiv h}$$

and so

$$(I - \mu S)(F_{\bar z}) = h$$

[here note: $S(\varphi_{\bar z}) = \varphi_z$ for $\varphi \in W^{1,q}(\mathbb{C})$ globally; thus need η to create a $W^{1,q}(\mathbb{C})$ - fcn out of f.]

Now

(7.4) $$F_{\bar z} = (I - \mu S)^{-1} h \in L^t(\mathbb{C}) \text{ for some } 2 < t < P(k).$$

Thus F (Hölder)-continuous, $F = \mathcal{C}(F_{\bar z}) \in \mathrm{Lip}_{(1-2/t)}$
$\Rightarrow$ f continuous, in fact locally Hölder continuous!

Further, $F_z = S(F_{\bar z}) \in L^t(\mathbb{C})$ and so

(7.5) $$\| DF \|_{L^t(\mathbb{C})} \le \|F_z\|_{L^t} + \|F_{\bar z}\|_{L^t} \le c\|h\|_{L^t} \le c \|f D\eta\|_{L^t(\mathbb{C})}$$

But now h Hölder cont & has compact support $\Rightarrow$ $h \in L^p(\mathbb{C})$ $\forall p$ $\Rightarrow$

Can use (7.4) as long as $t < P(k)$!!

i.e. $F_{\bar z} \in L^s(\mathbb{C})$ for every $s < P(k)$. And (7.5) $\Rightarrow$

$$(7.6) \quad \| DF \|_{L^s(\mathbb{C})} \leq C \| f D\eta \|_{L^s(\mathbb{C})}, \quad s < P(k).$$

Finally, by triangle inequality,

$$\| \eta Df \|_{L^s} = \| DF - f D\eta \|_{L^s(\mathbb{C})} \leq C \| f D\eta \|_{L^s(\mathbb{C})}. \square$$

What Caccioppoli inequality gives is an self-improving property for solutions to Beltrami equations: we start with regularity $f \in W^{1,q}_{loc}$ for some $Q(k) < q < 2$ and automatically end with better regularity $f \in W^{1,s}_{loc}$, $2 < s < P(k)$.

But this self-improved regularity is very delicate; it does not work below $Q(k)$:

7.3. Example Let $f(z) = \frac{1}{z} |z|^{1 - 1/K}$. Then

$$|\bar\partial f| = \frac{K-1}{K+1} |\partial f| \quad \& \quad f \in W^{1,q}_{loc}(\mathbb{C}) \quad \forall \, q < 1+k = \frac{2K}{K+1}$$

In fact, $f = \varphi \circ F$, $\varphi(z) = \frac{1}{z}$; $F(z) = z |z|^{\frac{1}{K} - 1}$ is K-qc $(K \geq 1)$.

Remark It can [AIM] be shown that Caccioppoli works for all $q \geq 1+k$.

Next, we need a version of chain rule, that works for quasiregular maps. In this connection it is convenient to introduce the Royden algebra

$$R(\Omega) = C(\Omega) \cap L^\infty(\Omega) \cap \mathring{W}^{1,2}(\Omega)$$

$R(\Omega)$ is indeed an algebra, with product $(fg)(x) = f(x)g(x)$ and norm

$$\|f\|_* = \|f\|_\infty + \|\nabla f\|_{L^2}$$

This makes $R(\Omega)$ a Banach algebra.

We have now the following form of chain rule:

7.4. Theorem Let $f : \Omega \to \Omega'$ be a K-quasiconformal homeo and let $u \in R(\Omega')$. Then $u \circ f \in R(\Omega)$ and

$$\nabla(u \circ f)(z) = Df(z)^t \, \nabla u(fz), \text{ i.e. } D(u \circ f)(z) = Du(fz) \circ Df(z)$$

Proof: see the enclosed copies, p. 76 A–B, and Thm 3.8.2 / AIM. $\square$

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Certainly, we may find simple functions $u_\nu \leqslant u$ converging almost everywhere to u, and for these u_ν we have

$$\int_\Omega u_\nu(f(z))J(z,f) = \int u_\nu(w) \to \int u(w)$$

The only problem we have to overcome then is to show that $u_\nu(f(z)) \to u(f(z))$ almost everywhere. This is clear from Theorem 3.7.5. □

The most convenient way to present the chain rule is in terms of the Royden algebra $R(\Omega)$ of a domain Ω. Recall that the elements of this algebra are the continuous and bounded functions with distributional derivatives in $L^2(\Omega)$,

$$R(\Omega) = C(\Omega) \cap L^\infty(\Omega) \cap \mathbb{W}^{1,2}(\Omega)$$

Here, and throughout this monograph, we use the notation

$$\mathbb{W}^{1,p}(\Omega) = \{v : \nabla v \in L^p(\Omega, \mathbb{C}^2)\}$$

for the homogeneous Sobolev space, that is, for the space of (locally integrable) functions with L^p-integrable gradient. No assumptions are made here on the L^p-integrability of the function itself.

For a bounded domain, however, $R(\Omega) \subset W^{1,2}(\Omega)$. The Royden algebra is a Banach algebra with the norm

$$\|v\|_* = \|v\|_\infty + \|\nabla v\|_2$$

According to the next result, quasiconformal mappings of Ω preserve the Royden algebra $R(\Omega)$. In fact, the reader may easily verify that a quasiconformal $f : \Omega \to \Omega'$ induces an algebra isomorphism $T_f : R(\Omega') \to R(\Omega)$, $T_f(v) = v \circ f$. Somewhat deeper lies the fact that every algebra isomorphism of the Royden algebra is of this type [230].

Theorem 3.8.2. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping and let $u \in R(\Omega')$. Then the composition $v = u \circ f$ lies in the Royden algebra $R(\Omega)$ with derivative*

$$\nabla v(z) = D^t f(z) \nabla u(f(z)), \quad \textit{almost everywhere in } \Omega \tag{3.39}$$

Further, we have the estimate

$$\int_\Omega |\nabla v(z)|^2 \, dz \leqslant K \int_{\Omega'} |\nabla u(z)|^2 \, dz \tag{3.40}$$

Proof. The main point is showing that $u \circ f$ has square-integrable distributional derivatives, and the argument is a variation of Lemma 3.5.2. We first assume that $u_\beta \in C^\infty(\overline{\Omega'})$ converge to u locally uniformly and in $\mathbb{W}^{1,2}(\Omega)$. From Lemma 3.5.2 we have $u_\beta \circ f \in \mathbb{W}^{1,2}(\Omega)$ and at the points of differentiability of f, hence almost everywhere,

$$\nabla(u_\beta \circ f)(z) = D^t f(z) \nabla u_\beta(f(z)),$$

(76B)

which we may write equivalently in the integral form

$$-\int_\Omega \nabla\varphi(z)\, u_\beta(f(z)) = \int_\Omega \varphi(z) D^t f(z) \nabla u_\beta(f(z)) \tag{3.41}$$

for every test funtion $\varphi \in C_0^\infty(\Omega)$.

Here passing to the limit as $\beta \to \infty$ on the left hand side poses no difficulty whatsoever as $u_\beta(f(z)) \to u(f(z))$ locally uniformly. However, we must justify passing to the limit on the right hand side. We do this as follows, using the fact that f is K-quasiconformal to give $|D^t f|^2 \leqslant KJ(z,f)$.

$$\begin{aligned}
&\int_\Omega |D^t f(z) \nabla u_\beta(f(z)) - D^t f(z) \nabla u(f(z))|^2 \\
&= \int_\Omega |D^t f(z)[\nabla u_\beta - \nabla u](f(z))|^2 \\
&\leqslant K \int_\Omega J(z,f)|[\nabla u_\beta - \nabla u](f(z))|^2 \\
&= K \int_{\Omega'} |\nabla u_\beta - \nabla u|^2
\end{aligned}$$

by the change-of-variable formula (3.38). Now,

$$\int_{\Omega'} |\nabla u_\beta - \nabla u|^2 \to 0 \quad \text{as } \beta \to \infty$$

We end up with the identity

$$-\int_\Omega \nabla\varphi(z)\, u(f(z)) = \int_\Omega \varphi(z) D^t f(z) \nabla u(f(z)) \tag{3.42}$$

for every $C_0^\infty(\Omega)$-test function. Directly from the definition of the Sobolev space we see that $v(z) = u(f(z))$ lies in the algebra $W^{1,2}(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$, and its gradient is as given at (3.39). The integral inequality is immediate from the change-of-variable formula of Theorem 3.8.1. □

3.9 Quasisymmetry and Equicontinuity

Next, we set about proving Theorem 3.1.3. We first establish the result for quasisymmetric mappings, which will in turn establish the desired result. First, let us prove that a family of normalized quasisymmetric mappings is equicontinuous. We shall use this fact a bit later on in the book. We draw the reader's attention to Section 2.9.4 where the definition of equicontinuity is given.

Theorem 3.9.1. *Let $A \subset \mathbb{C}$ with $0, 1 \in A$. Then the family of all η-quasisymmetric maps $f : A \to \mathbb{C}$ with $f(0) = 0$ and $f(1) = 1$ is equicontinuous.*

Proof. Let $a_0 \in A$. Then for any $a \in A$,

$$\frac{|f(a_0) - f(a)|}{|f(a_0) - f(1)|} \leqslant \eta\left(\frac{|a_0 - a|}{|a_0 - 1|}\right) \quad \text{and} \quad \frac{|f(a_0) - f(a)|}{|f(a_0) - f(0)|} \leqslant \eta\left(\frac{|a_0 - a|}{|a_0|}\right)$$

With these tools we can classify all $W^{1,2}_{loc}$-solutions to the Beltrami eq.

7.5. Theorem (Stoilow Factorization) Let $h \in W^{1,2}_{loc}(\Omega)$ be (some) solution to

$$(7.7) \qquad \bar\partial h = \mu(z)\, \partial h \quad \text{a.e. } z \in \Omega; \quad |\mu(z)| \le k < 1 \quad \text{a.e. } z.$$

Then, if $f : \Omega \to \Omega'$ is a $W^{1,2}_{loc}$-homeo solving (7.7), we have

$$h(z) = \Phi \circ f(z) \qquad , z \in \Omega,$$

where Φ analytic in $\Omega' = f(\Omega)$.

Proof: Define $\Phi = h \circ f^{-1}$. Then h and thus Φ continuous by Thm 7.2. By restricting to a subdomain have $\Phi \in L^\infty(\Omega)$ and $h \in R(\Omega) \overset{\text{Thm 7.4}}{\Rightarrow} \Phi \in R(\Omega)$. As Thm 7.4 gives pointwise chain rule, $D\Phi = Dh(f^{-1}) \circ Df^{-1}$, i.e.

$$\bar\partial \Phi(f_z) = h_z \cdot (f^{-1})_{\bar z}(f) + h_{\bar z} \cdot \overline{(f^{-1})_z(f)}$$

But the chain rule formulae from p 52 (with $g = f^{-1}$) give

$$(f^{-1})_{\bar z}(f) = -\frac{f_{\bar z}(z)}{J(z,f)} \overset{(7.7)}{=} -\frac{\mu(z) f_z(z)}{J(z,f)}$$

and

$$\overline{(f^{-1})_z}(f) = \frac{f_z(z)}{J(z,f)}$$

$$\Rightarrow \bar\partial \Phi(f(z)) \cdot J(z,f) = -h_z\, \mu(z)\, f_z + h_{\bar z}(z) f_z(z) \overset{(7.7)}{=} 0 \text{ a.e.}$$

As the q conf f has Lusin properties by Thm. 4.1, it follows $\bar\partial \Phi = 0$ a.e., and as $\Phi \in R(\Omega') \subset W^{1,2}_{loc}(\Omega')$, Weyl's lemma proves Φ analytic. $\square$

<u>Remarks</u> a) M.R.M.T gives homeomorphic solutions to (7.7), just set $\mu_0(z) = \begin{cases} \mu(z), & z \in \Omega \\ 0, & z \in \mathbb{C}\setminus\Omega \end{cases}$, let F be the $W^{1,2}_{loc}$-homeo sol'n to $\bar\partial F = \mu_0 \partial F$, and take $f = F|_{\Omega}$.

b) If Ω simply connected, let $\varphi : \Omega' = f(\Omega) \to \Omega$ be a conformal map. Then $\tilde f = \varphi \circ f$ solves the <u>same</u> Beltrami equation (7.7). Thus can <u>always</u> find a homeomorphic $W^{1,2}_{loc}(\Omega)$-solution to (7.7) with

$$f : \Omega \to \Omega.$$

c) Conversely to Thm 7.5, if Φ analytic in $\Omega' = f(\Omega)$ then $h := \Phi \circ f$ solves (7.7). Thus

$$\boxed{h \text{ a } W^{1,2}_{loc}\text{-sol. to (7.7)} \iff h = \Phi \circ f, \quad \begin{array}{l} f \text{ q conf} \\ \Phi \text{ analytic} \end{array}}$$

classifies all solution to the Beltrami eq'n.

We conclude with an application of Stoïlow factorization to the structure of solutions to PDE's.

Assume

(7.8a) Ω is simply connected

(7.8.b) $\sigma : \Omega \to \mathbb{R}^{2\times 2}$ with $\sigma(z)^t = \sigma(z)$ a.e. z

(7.8.c) $\frac{1}{K}|h|^2 \le \langle h, \sigma(z) h \rangle \le K|h|^2$

<u>7.6. Corollary</u> Suppose $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution to

$$\nabla \cdot \sigma(z) \nabla u = 0 \qquad \text{in } \Omega.$$

Then

$$u = w \circ f$$

where $\Delta w = 0$ (i.e. w <u>harmonic</u> $\Rightarrow C^\infty$ etc.) and f is K-quasiconformal (with <u>same</u> K as in (7.8.c))

<u>Remarks</u>: a) The factorization $u = w \circ f$ gives a complete understanding of the properties of u, e.g. u sat. maximum principle (as w does), $u \in Lip_{1/K}$ (as f does) etc. etc.

b) The factorization works for $W^{1,q}_{loc}$-solutions, <u>as long as</u> they satisfy the Caccioppoli inequalities – these give automatic improvement from $W^{1,q}_{loc}$ to $W^{1,2}_{loc}$

c) Conversely, if $u = w \circ f$ with $\Delta w = 0$ and f K-q conf, then

$\nabla \cdot \sigma \nabla u = 0$, where $\sigma(z)$ satisfies (7.8. a–c);

Here use $\bar\partial f = \mu_0 \partial f$ with Thm. 5.5. (again same K)

Proof of Corollary 7.6.:

Let $\nabla v = * \sigma(z) \nabla u$ be the σ-harmonic conjugate.

Then by Thm. 5.5. we know $h := u + iv$ satisfies

$$h_{\bar z} = \mu h_z + \nu \overline{h_z} .$$

Here we need to determine the ellipticity,

$$|\mu(z)| + |\nu(z)| \le ?$$

Given the point z_0, $\sigma(z_0)^t = \sigma(z_0) \Rightarrow$ we can conjugate $\sigma(z_0)$ by a rotation so that $\sigma(z_0)$ becomes diagonal; this conjugation does not change (7.8.c) or the value of $|\mu(z_0)| + |\nu(z_0)|$. (why?)

Now if $\sigma(z_0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ (ass. $\lambda_1 \le \lambda_2$), then (7.8) $\Rightarrow \frac{1}{K} \le \lambda_1 \le \lambda_2 \le K$,

and by (5.6.c),

$$\mu(z_0) = \frac{\lambda_2 - \lambda_1}{(1+\lambda_1)(1+\lambda_2)}, \quad \nu = \frac{1 - \lambda_1 \lambda_2}{(1+\lambda_1)(1+\lambda_2)}$$

If $\lambda_2 < 1/\lambda_1 \Rightarrow |\mu| + |\nu| = \frac{\lambda_2 - \lambda_1 + (1 - \lambda_1 \lambda_2)}{(1+\lambda_1)(1+\lambda_2)} = \frac{1-\lambda_1}{1+\lambda_1} \cdot \cancel{\frac{1+\lambda_2}{1+\lambda_2}}$

As $\frac{1}{K} \le \lambda_1 \le K$ this gives $|\mu| + |\nu| \le \frac{K-1}{K+1} =: k$.

Similarly if $\lambda_2 > 1/\lambda_1$. We thus know that

$$|h_{\bar z}| \le k |h_z| \quad \text{or} \quad h_{\bar z} = \tilde\mu\, h_z, \quad |\tilde\mu| \le k.$$

By Stoilow, $h = \Phi \circ f$, where f is K-quasiconformal (recall (5.2)/p. 43), and where Φ analytic. Thus $u = \mathrm{Re}\, h = \underbrace{(\mathrm{Re}\,\Phi)}_{\equiv w \text{ harmonic}} \circ f$. $\square$

VIII Holomorphic Motions

Holomorphic motions give yet another approach — completely different from the previous ones — to quasiconformal mappings.

Let us first return to the proof of M.R.M.T. with $|\mu| \le k\, \chi_{B(0,R)}$, for some $R < \infty$.
If f is the principal solution to $\bar\partial f = \mu \partial f$, then

$$f(z) = z + \mathcal{C}\big((I - \mu S)^{-1}\mu\big) \tag{8.1}$$ (c.f. p. 62)

which shows that f depends holomorphically on μ!
This can be interpreted in many ways; we use the following:

Define $\mu_\lambda^{(z)} = \frac{\lambda}{k}\mu(z)$, $|\lambda| < 1$.

Then $\|\mu_\lambda\|_\infty \le |\lambda| < 1$ and μ_λ has compact support $\Rightarrow$

Can solve

(8.2) $$\bar\partial f = \mu_\lambda \partial f \;; \quad f = f^\lambda(z) = z + \mathcal{O}(1/z) \text{ at } \infty.$$

To make better use of representation (8.1) here, we apply the L^p-theory:

Let $g < 1$ and fix $2 < p < P(g)$. (i.e. $g\|S\|_p < 1$)

Then
$$|\lambda| < g \Rightarrow \|\mu_\lambda\|_\infty \|S\|_p \le g\|S\|_p < 1 \Rightarrow$$

$\sum_{n=0}^{\infty} (\mu_\lambda S)^n \mu_\lambda \in L^p(\mathbb{C})$, where sum converges absolutely,

$\sum \|(\mu_\lambda S)^n \mu_\lambda\|_{L^p} < \infty$. For each fixed z, $\frac{1}{\zeta - z} \in L^q(B(0,R))$,

$\frac{1}{p} + \frac{1}{q} = 1$ and therefore, for any $z \in \mathbb{C}$, ($q < 2$!)

$$f^\lambda(z) = z + \mathcal{C}\Big(\sum_{n=0}^{\infty} (\mu_\lambda S)^n \mu_\lambda\Big)(z)$$

$$= z + \sum_{n=0}^{\infty} \mathcal{C}\big((\mu_\lambda S)^n \mu_\lambda\big)(z) = z + \sum_{n=0}^{\infty} \lambda^n \mathcal{C}\Big[\Big(\frac{\mu}{k} S\Big)^n \frac{\mu}{k}\Big](z)$$

(one can use also Exercise set 3 to get this representation).

Thus we see that for any fixed $z \in \mathbb{C}$, $f^\lambda(z)$ can be represented as a power series in λ, thus $\lambda \mapsto f^\lambda(z)$ is analytic in λ, $|\lambda| < g$. But $g < 1$ arbitrary $\Rightarrow$ get

<u>8.1. Theorem</u> Let f be the principal solution to $\bar\partial f = \mu\,\partial f$ and let f^λ solve (8.2) with $\mu_\lambda = \frac{\lambda}{k}\mu$. Then

a) $f^k = f$

b) $\lambda \mapsto f^\lambda(z)$ analytic in $\mathbb{D}$, for any fixed $z \in \mathbb{C}$.

c) $z \mapsto f^\lambda(z)$ homeo in $\mathbb{C}$, for any fixed λ.

d) $f^0(z) \equiv z$

<u>Proof</u> a), d) follow from uniqueness of principal sol's and b) was explained above & c) on p. 71 – 72. $\square$

Thus we have embedded any (principal) quasiconf. map $f : \mathbb{C} \to \mathbb{C}$ to a holomorphic flow of qc maps!

<u>Remark</u> Often principal solution is not the best normalization. If we set

$$(8.3) \qquad F^\lambda(z) = \frac{f^\lambda(z) - f^\lambda(0)}{f^\lambda(1) - f^\lambda(0)} \quad \Rightarrow \quad F^\lambda(0)=0,\ F^\lambda(1)=1$$

and a) – d) still hold for F^λ ($F^k = F$, the normalized solution to $\bar\partial F = \mu\,\partial F$) !!

<u>Question</u> Can we take the limit $R \to \infty$ and keep all good properties a) – d) of Theorem 8.1 ?